Mathematical Physics I (Fall 2021): Final Exam Solution

Dec. 11, 2021

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. (a) [2 pt] Write and solve the Euler equation to make the following integral stationary. Change the independent variable, if needed, to make the Euler equation simpler.

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{1+y} dx$$

(b) [2 pt] A curve $y = y(x)$ joining two points x_1 and x_2 on the x-axis, is revolved around the x-axis to produce a surface and a volume of revolution. Given the volume, find the shape of the curve $y = y(x)$ that minimizes the surface area. (Note: Once you find a first integral of the Euler equation, you may want to first determine the constant of integration using the fact that $y = 0$ at the endpoints.)

• (a) As in Example 3 of Boas Chapter 9, Section 3, with $x' = \frac{dx}{dy} = \frac{1}{y'}$,

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{1+y} dx = \int_{y_1}^{y_2} \frac{\sqrt{1+y'^2}}{1+y} x' dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2+1}}{1+y} dy.$$

Now, the Euler equation becomes

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x'} = 0 = \frac{\partial}{\partial y}\left(\frac{x'}{(1+y)\sqrt{x'^2+1}}\right) \;\rightarrow\; \frac{x'^2}{(1+y)^2(x'^2+1)} = C_1^2 \;\rightarrow\; x' = \frac{C_1(1+y)}{\sqrt{1-C_1^2(1+y)^2}}.$$

Integrating both sides

$$x = \int \frac{C_1(1+y)dy}{\sqrt{1-C_1^2(1+y)^2}} = -\frac{1}{C_1}\sqrt{1-C_1^2(1+y)^2} + C_2 = -\sqrt{\frac{1}{C_1^2} - (1+y)^2} + C_2,$$

from which you acquire $(x - C_2)^2 + (1+y)^2 = (\frac{1}{C_1})^2$.

• (b) As in Example 1 of Boas Chapter 9, Section 6, with $A = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}dx$ and $V = \int_{x_1}^{x_2} \pi y^2 dx$, the integral we want to minimize is

$$I = \pi\int_{x_1}^{x_2} (2y\sqrt{1+y'^2} + \lambda y^2)dx = \int_{y_1}^{y_2} (2y\sqrt{1+y'^2} + \lambda y^2)x' dy = \int_{y_1}^{y_2} (2y\sqrt{x'^2+1} + \lambda x' y^2)dy.$$

where λ is the Lagrange multiplier, and we changed the independent variable from x to y as in (a). Now, the Euler equation becomes

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x'} = 0 = \frac{\partial}{\partial y}\left(\frac{2yx'}{\sqrt{x'^2+1}} + \lambda y^2\right) \;\rightarrow\; \frac{2yx'}{\sqrt{x'^2+1}} + \lambda y^2 = C_1 = 0 \;\rightarrow\; x' = \frac{y}{\sqrt{\left(\frac{2}{\lambda}\right)^2 - y^2}},$$

in which the integration constant C_1 is determined with $y = 0$ at the endpoints. Integrating both sides

$$x = \int \frac{ydy}{\sqrt{\left(\frac{2}{\lambda}\right)^2 - y^2}} = -\sqrt{\left(\frac{2}{\lambda}\right)^2 - y^2} + C_2,$$

from which you acquire $(x - C_2)^2 + y^2 = (\frac{2}{\lambda})^2$.

2. Find the general solution to each of the following differential equations.

(a) [1 pt] $y' + y\cos x = \sin 2x$

(b) [2 pt] $y'' + 4y' + 5y = 26e^{3x} + 2e^{-2x}\cos x$

• (a) From Eqs.(3.4) and (3.9) of Boas Chapter 8, $I = \int P dx = \int \cos x dx = \sin x \;\rightarrow\; y = e^{-I}\int Qe^{I}dx + Ce^{-I} = e^{-\sin x}\int \sin 2x \cdot e^{\sin x}dx + Ce^{-\sin x}$. Here we use change the variable to get $\int \sin 2x \cdot e^{\sin x}dx = 2\int \sin x\cos x \cdot e^{\sin x}dx = 2\int ue^{u}du = 2(ue^{u} - e^{u}) = 2e^{\sin x}(\sin x - 1)$. Therefore, $y = e^{-\sin x}\int \sin 2x \cdot e^{\sin x}dx + Ce^{-\sin x} = 2(\sin x - 1) + Ce^{-\sin x}$.

• (b-1) The homogeneous equation $y'' + 4y' + 5y = 0$ gives the complementary solution $y_c = e^{-2x}(A\sin x + B\cos x)$. And the inhomogeneous equation $y'' + 4y' + 5y = 26e^{3x}$ has the particular solution $y_{p1} = e^{3x}$.

• (b-2) Now to find the particular solution y_{p2} for $y'' + 4y' + 5y = 2e^{-2x}\cos x$, as in Example 6 of Boas Chapter 8, Section 6, we study $Y'' + 4Y' + 5Y = 2e^{(-2+i)x}$. Because $-2 + i$ equals to one of the roots of the auxiliary equation, from Eq.(6.18) of Boas Chapter 8 you try the particular solution of the form $Y_{p2} = Cxe^{(-2+i)x}$. Plugging this trial solution into the differential equation of Y, you get $2Cie^{(-2+i)x} = 2e^{(-2+i)x}$, which leads to $C = -i$ and $Y_{p2} = -ixe^{(-2+i)x}$. Therefore, the particular solution we needed is $y_{p2} = Re(Y_{p2}) = xe^{-2x}\sin x$. Combining all the above, you reach $y = y_c + y_{p1} + y_{p2} = e^{-2x}(A\sin x + B\cos x) + e^{3x} + xe^{-2x}\sin x$.

3. (a) [2 pt] Consider a ball of mass m falling under the gravitational acceleration g. The ball experiences a resistive force proportional to its speed (with the constant of proportionality b). Using y as the distance the ball traveled (i.e., $y > 0$), find its differential equation of motion. Then find the speed of the ball at time t, $y'(t)$, with the initial condition $y'(0) = 0$. To receive the full 2 points, you need to solve this problem in two ways: (*i*) by directly solving the 1st-order ordinary differential equation, and (*ii*) by using the Laplace transform. (Note: The table of Laplace transforms is in the last page of this exam.)

(b) [2 pt] Now consider a different type of falling objects. Think of a water droplet falling through a mist cloud, *increasing in mass* as it picks up moisture. As the droplet passes through the cloud, it remains spherical in shape (radius r), but acquires mass at a rate proportional to its cross-sectional area πr^2 multiplied by the speed of the fall (i.e., the mass increase as it travels dy is proportional to the volume $\pi r^2 dy$ swept out by the droplet during dt). The droplet does not experience any resistive force. Find the differential equation of motion of the droplet, assuming it starts from rest with an infinitely small size, $y(0) = y'(0) = r(0) = 0$.

(c) [1 pt] Show that the droplet's acceleration of the fall is $g/7$. (Note: Once you have the equation of motion, you may assume that y'^2 can be written as $y'^2 = \sum_n a_n y^n$, or more simply for this case, $y'^2 = Cy^n$. You may also want to prove and utilize the relation $y'' = \frac{1}{2}\frac{d(y'^2)}{dy}$.)

- (a-1) The equation of motion $F = my'' = mg - by'$ (with both $\hat{\mathbf{e}}_y$ and $\mathbf{g}$ pointing downwards) $\rightarrow \frac{dy'}{g - \frac{b}{m}y'} = dt \rightarrow -\frac{m}{b}\ln\left(g - \frac{b}{m}y'\right) = t + C_1 \rightarrow y'(t) = \frac{mg}{b} - \frac{mC_2}{b}e^{-\frac{b}{m}t} = \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right)$.

- (a-2) Alternatively, taking the Laplace transform of the equation with $Y' = L(y')$ and $y'(0) = 0$, you find $mpY' = \frac{mg}{p} - bY' \rightarrow Y' = \frac{g}{p(p+\frac{b}{m})} = \frac{mg}{b}\left(\frac{1}{p} - \frac{1}{p+\frac{b}{m}}\right) \rightarrow y' = \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right)$.

- (b) From $m' = k\pi r^2 y' = \rho\, 4\pi r^2 r'$, you can see $r = \frac{k}{4\rho}y$ since $r(0) = 0$. That is, the radius $r(t)$ of the droplet increases as $y(t)$. Then, the equation of motion $F = my'' + m'y' = mg \rightarrow (\rho\frac{4}{3}\pi r^3)y'' + (k\pi r^2 y')y' = (\rho\frac{4}{3}\pi r^3)g \rightarrow y'' + \frac{y'^2}{(4\rho/3k)r} = y'' + \frac{y'^2}{(4\rho/3k)((k/4\rho)y)} = y'' + \frac{3y'^2}{y} = g$.

- (c-1) With $y'^2 = Cy^n$, the equation of motion becomes $y'' + \frac{3y'^2}{y} = g \rightarrow \frac{1}{2}Cny^{n-1} + \frac{3Cy^n}{y} = (\frac{n}{2} + 3)Cy^{n-1} = g$. Equating the coefficient you get $n = 1$ and $C = \frac{2}{7}$. Therefore, $y'^2 = \frac{2}{7}gy$.

- (c-2) Alternatively, with $y'^2 = \sum_n a_n y^n$, the equation of motion becomes $y'' + \frac{3y'^2}{y} = g \rightarrow \sum_n \frac{1}{2}na_n y^{n-1} + \sum_n 3a_n y^{n-1} = g$. (Here we used $y'' = \frac{dy'}{dy}\cdot\frac{dy}{dt} = \frac{dy'}{dy}y' = \frac{1}{2}\frac{d(y'^2)}{dy}$.) Equating the coefficients you get $\frac{1}{2}na_n + 3a_n = g$ only if $n = 1$. Therefore, the only nonzero coefficient is $a_1 = \frac{2}{7}g$, which means $y'^2 = \frac{2}{7}gy$.

4. (a) [1 pt] Show that the Dirac delta function $\delta(x)$ expanded in a Fourier series in the interval $(-\pi, \pi)$ is written as

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx.$$

(b) [1 pt] By writing $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, prove that

$$\delta_N(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos nx = \frac{1}{2\pi}\frac{\sin\,(N+\frac{1}{2})x}{\sin\frac{1}{2}x}.$$

This is one of many examples of sequences of functions $\delta_N(x)$ discussed in the class and in the homework, the limiting cases of which are used to develop the properties of $\delta(x)$.

- (a) From Eqs.(5.9)-(5.10) of Boas Chapter 7, you find $b_n = 0$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)dx = \frac{1}{\pi},$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos nx\, dx = \frac{1}{\pi}\cos 0 = \frac{1}{\pi},$$

in which we also utilized Eq.(11.6) of Boas Chapter 8. Thus, we acquire

$$\delta(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx.$$

- (b) The sum of complex geometric series is evaluated as

$$\begin{aligned}\sum_{n=1}^{N}\cos nx &= \frac{1}{2}\sum_{n=1}^{N}(e^{inx} + e^{-inx}) = \frac{1}{2}\left[\frac{1-e^{i(N+1)x}}{1-e^{ix}} + \frac{1-e^{-i(N+1)x}}{1-e^{-ix}}\right]\\ &= \frac{1}{2}\left[\frac{e^{-i\frac{x}{2}} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} + \frac{e^{i\frac{x}{2}} - e^{-i(N+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}\right]\\ &= \frac{1}{2}\left[\frac{-e^{-i\frac{x}{2}} + e^{i(N+\frac{1}{2})x} + e^{i\frac{x}{2}} - e^{-i(N+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}\right] = \frac{1}{2}\left[-1 + \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\right].\end{aligned}$$

Therefore,

$$\delta_N(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos nx = \frac{1}{2\pi} + \frac{1}{2\pi}\left[-1 + \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}\right] = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}.$$

5. (a) [2 pt] The periodic function $f(x) = x \quad (-\pi < x < \pi)$ has a period 2π. Sketch several periods of this function. Expand $f(x)$ in a sine-cosine Fourier series, and use your result with Dirichlet's theorem and Parseval's theorem to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}, \quad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}, \quad \text{and} \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}.$$

(Note: For the last equality, you may want to consider integrating the Fourier series of $f(x)$ term by term in the interval $[0, x]$ where $0 < x \le \pi$.)

(b) [2 pt] Find the exponential Fourier transform of

$$f(x) = \begin{cases} \cos x, & \text{if } |x| < \frac{\pi}{2}, \\ 0, & \text{if } |x| > \frac{\pi}{2}, \end{cases}$$

and use your result with the Fourier integral theorem and Parseval's theorem to evaluate

$$\int_0^{\infty} \frac{\cos(\pi\alpha/2)}{1-\alpha^2}\, d\alpha \quad \text{and} \quad \int_0^{\infty} \frac{\cos^2(\pi\alpha/2)}{(1-\alpha^2)^2}\, d\alpha.$$

• (a) From Eq.(9.4) of Boas Chapter 7, you can find $a_n = 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{-2(-1)^n}{n},$$

which gives

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n} = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right).$$

From Dirichlet's theorem in Boas Chapter 7, Section 6, you reach

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right),$$

and from Parseval's theorem in Eq.(11.4) of Boas Chapter 7, you also find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{\pi^2}{3} = \frac{1}{2}\sum_{n=1}^{\infty}b_n^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2} = 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right).$$

Lastly, integrating the Fourier series of $f(x)$ term by term in the interval $[0, x]$ yields

$$\int_0^x f(x)dx = \frac{x^2}{2} = 2\left[\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}\right]_0^x = 2\left\{\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right\},$$

in which you plug $x = \pi$ to get

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

Therefore,

$$-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots = \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{\pi^2}{12}.$$

• (b) From Eq.(12.2) of Boas Chapter 7, you can find

$$\begin{aligned} g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\cos x\,e^{-i\alpha x}\,dx = \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}(e^{ix} + e^{-ix})\,e^{-i\alpha x}\,dx \\ &= \frac{1}{2\pi}\left[\frac{\sin((1+\alpha)\pi/2)}{1+\alpha} + \frac{\sin((1-\alpha)\pi/2)}{1-\alpha}\right] = \frac{1}{\pi}\left[\frac{\cos(\pi\alpha/2)}{1-\alpha^2}\right], \end{aligned}$$

which gives

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi\alpha/2)}{1-\alpha^2}e^{i\alpha x}\,d\alpha.$$

From the Fourier integral theorem in Boas Chapter 7, Section 12, you reach

$$1 = f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi\alpha/2)}{1-\alpha^2}\,d\alpha = \frac{2}{\pi}\int_0^{\infty}\frac{\cos(\pi\alpha/2)}{1-\alpha^2}\,d\alpha,$$

and from Parseval's theorem in Eq.(12.24) of Boas Chapter 7, you also find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx = \frac{1}{4} = \int_{-\infty}^{\infty}|g(\alpha)|^2\,d\alpha = \frac{2}{\pi^2}\int_0^{\infty}\frac{\cos^2(\pi\alpha/2)}{(1-\alpha^2)^2}\,d\alpha.$$

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in $\sim$3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

• (a) See the student presentation slides in Lecture 15-1 that include the collection of term project presentations by five students on December 6.

• (b) See the class slides for Lecture 14-2 that include many example problems, and the grading guideline.