Mathematical Physics I (Fall 2021): Final Examination

Dec. 11, 2021

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from ① to ⑥ on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet ②). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

• Make your writing easy to read. Illegible answers will *not* be graded.

1. (a) [2 pt] Write and solve the Euler equation to make the following integral stationary. Change the independent variable, if needed, to make the Euler equation simpler.

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{1+y} dx$$

(b) [2 pt] A curve $y = y(x)$ joining two points x_1 and x_2 on the x-axis, is revolved around the x-axis to produce a surface and a volume of revolution. Given the volume, find the shape of the curve $y = y(x)$ that minimizes the surface area. (Note: Once you find a first integral of the Euler equation, you may want to first determine the constant of integration using the fact that $y = 0$ at the endpoints.)

2. Find the general solution to each of the following differential equations.

(a) [1 pt] $y' + y\cos x = \sin 2x$

(b) [2 pt] $y'' + 4y' + 5y = 26e^{3x} + 2e^{-2x}\cos x$

3. (a) [2 pt] Consider a ball of mass m falling under the gravitational acceleration g. The ball experiences a resistive force proportional to its speed (with the constant of proportionality b). Using y as the distance the ball traveled (i.e., $y > 0$), find its differential equation of motion. Then find the speed of the ball at time t, $y'(t)$, with the initial condition $y'(0) = 0$. To receive the full 2 points, you need to solve this problem in two ways: (i) by directly solving the 1st-order ordinary differential equation, and (ii) by using the Laplace transform. (Note: The table of Laplace transforms is in the last page of this exam.)

(b) [2 pt] Now consider a different type of falling objects. Think of a water droplet falling through a mist cloud, *increasing in mass* as it picks up moisture. As the droplet passes through the cloud, it remains spherical in shape (radius r), but acquires mass at a rate proportional to its cross-sectional area πr^2 multiplied by the speed of the fall (i.e., the mass increase as it travels dy is proportional to the volume $\pi r^2 dy$ swept out by the droplet during dt). The droplet does not experience any resistive force. Find the differential equation of motion of the droplet, assuming it starts from rest with an infinitely small size, $y(0) = y'(0) = r(0) = 0$.

(c) [1 pt] Show that the droplet's acceleration of the fall is $g/7$. (Note: Once you have the equation of motion, you may assume that y'^2 can be written as $y'^2 = \sum_n a_n y^n$, or more simply for this case, $y'^2 = Cy^n$. You may also want to prove and utilize the relation $y'' = \frac{1}{2}\frac{d(y'^2)}{dy}$.)

4. (a) [1 pt] Show that the Dirac delta function $\delta(x)$ expanded in a Fourier series in the interval $(-\pi, \pi)$ is written as

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos nx.$$

(b) [1 pt] By writing $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$, prove that

$$\delta_N(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N} \cos nx = \frac{1}{2\pi}\frac{\sin{(N+\frac{1}{2})x}}{\sin\frac{1}{2}x}.$$

This is one of many examples of sequences of functions $\delta_N(x)$ discussed in the class and in the homework, the limiting cases of which are used to develop the properties of $\delta(x)$.

5. (a) [2 pt] The periodic function $f(x) = x \quad (-\pi < x < \pi)$ has a period 2π. Sketch several periods of this function. Expand $f(x)$ in a sine-cosine Fourier series, and use your result with Dirichlet's theorem and Parseval's theorem to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}, \quad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}, \quad \text{and} \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}.$$

(Note: For the last equality, you may want to consider integrating the Fourier series of $f(x)$ term by term in the interval $[0, x]$ where $0 < x \leq \pi$.)

(b) [2 pt] Find the exponential Fourier transform of

$$f(x) = \begin{cases} \cos x, & \text{if } |x| < \frac{\pi}{2}, \\ 0, & \text{if } |x| > \frac{\pi}{2}, \end{cases}$$

and use your result with the Fourier integral theorem and Parseval's theorem to evaluate

$$\int_0^\infty \frac{\cos(\pi\alpha/2)}{1-\alpha^2}\, d\alpha \quad \text{and} \quad \int_0^\infty \frac{\cos^2(\pi\alpha/2)}{(1-\alpha^2)^2}\, d\alpha.$$

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in $\sim$3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.