Classical Mechanics II (Fall 2020): Homework #3

Due Nov. 17, 2020

[0.5 pt each, total 6 pts]

1. Thornton & Marion, Problem 12-6

(Note: For Problem 12-6, explain how one of the normal modes is damped but the other is not.)

• By plugging a trial solution in the form of Eq.(12.2), $x_1 = B_1 e^{i\omega t}$ and $x_2 = B_2 e^{i\omega t}$, in the equations of motion, $m\ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + \kappa x_1 = 0$ and $m\ddot{x}_2 + \beta(\dot{x}_2 - \dot{x}_1) + \kappa x_2 = 0$, and requiring nontrivial solution, we find $\omega^2 = \kappa/m$ and $\omega^2 = -(-\beta \pm \sqrt{\beta^2 - m\kappa})^2/m^2$.

2. Thornton & Marion, Problem 12-11

• By plugging a trial solution in the form of Eq.(12.2), $I_1 = B_1 e^{i\omega t}$ and $I_2 = B_2 e^{i\omega t}$, in the Kirchhoff circuit equations, $L\ddot{I}_1 + I_1/C + M\ddot{I}_2 = 0$ and $L\ddot{I}_2 + I_2/C + M\ddot{I}_1 = 0$, and requiring nontrivial solution, two eigenfrequencies appear, $\omega = \sqrt{\frac{1}{C(L\pm M)}}$.

3. Thornton & Marion, Problem 12-15

(Note: For Problem 12-15, discuss how your equations of motion differ from those in Problem 12-13. Discuss also how the circuit in this problem is the "equivalent electrical circuit" of the system in Problem 12-6 — a term explained in Section 3.7 of Thornton & Marion — if $L_1 = L_2 = L$ and $C_1 = C_2 = C$.)

• Follow the similar procedure as Problem 12-11, but with a different set of Kirchhoff equations, $L_1\ddot{I}_1 + R(\dot{I}_1 - \dot{I}_2) + I_1/C_1 = 0$ and $L_2\ddot{I}_2 + R(\dot{I}_2 - \dot{I}_1) + I_2/C_2 = 0$.

• For Problem 12-13, the set of Kirchhoff equations are $(L_1 + L_{12})\ddot{I}_1 + I_1/C_1 - L_{12}\ddot{I}_2 = 0$ and $(L_2 + L_{12})\ddot{I}_2 + I_2/C_2 - L_{12}\ddot{I}_1 = 0$.

4. Thornton & Marion, Problem 12-21

• $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$, $U = \frac{1}{2}\left[\kappa_1(x_1^2 + x_3^2) + \kappa_2 x_2^2 + \kappa_3(x_1x_2 + x_2x_3)\right]$. The eigenfrequencies are therefore $\omega_1 = 0$ (pure translation), $\omega_2 = \sqrt{\frac{\kappa_1}{m}}$ and $\omega_3 = \sqrt{\frac{\kappa_1+\kappa_2}{m}}$.

5. Thornton & Marion, Problem 12-22

(Note: For Problem 12-22, you may want to explain or justify why you do not have to consider the gravitational potential energy of the plate, when x_3 is defined as a displacement from the equilibrium position.)

• $T = \frac{1}{2}M\left(\dot{x}_3^2 + \frac{1}{3}A^2\dot{\theta}^2 + \frac{1}{3}B^2\dot{\phi}^2\right)$, $U = \frac{1}{2}\kappa\left(4x_3^2 + 4A^2\theta^2 + 4B^2\phi^2\right)$ (quadratic terms). Once a thin bar is added to the plate, T changes to $\frac{1}{2}\left[M\dot{x_3}^2 + \frac{1}{3}(M+m)A^2\dot{\theta}^2 + \frac{1}{3}MB^2\dot{\phi}^2\right]$ breaking the degeneracy in the eigenfrequencies.

6. Thornton & Marion, Problem 12-27

• $T = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2$, $U = \frac{1}{2}g\left[(m_1+m_2)L_1\theta_1^2 + m_2L_2\theta_2^2\right]$.

7. Thornton & Marion, Problem 13-3

• You can plot the first three terms of Eq.(13.13) with $t = 0$ and $t = \frac{T}{8}$. That is, $\left(\frac{\pi^2}{8h}\right)q(x,t) \simeq \sin\left(\frac{\pi x}{L}\right)\cos\omega_1 t - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right)\cos\omega_3 t + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right)\cos\omega_5 t$.

8. Thornton & Marion, Problem 13-6

(Notes: For Problem 13-6, refer to Example 13.1 for more explanation about qunatifying the energy in each of the excited modes – the fundamental and the harmonics.)

• From Eqs.(13.5) to (13.8), $\omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}}$, $\mu_r = 0$, and

$$\nu_r = -\frac{2}{\omega_r L}\left(\int_0^{\frac{L}{4}} \frac{4v_0 x}{L}\sin\frac{r\pi x}{L}dx + \int_{\frac{L}{4}}^{\frac{L}{2}}\left(2v_0 - \frac{4v_0 x}{L}\right)\sin\frac{r\pi x}{L}dx\right).$$

One can observe that ν_r vanishes with $r = 4n$ where n is an integer.

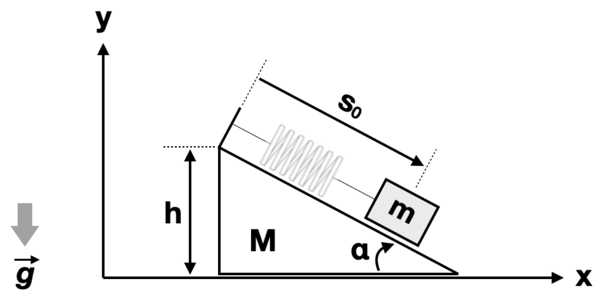
9. A block of negligible size and mass m is attached to a wedge of mass M by a massless spring of original unstretched length l and spring constant κ, as shown in the figure. The wedge's inclined surface makes an angle α with the horizontal. All surfaces are frictionless. g is the gravitational acceleration.

(a) When both the block and the wedge are at rest, find the equilibrium position of the block along the inclined surface, s_0 (see the figure).

(b) Assuming small oscillations, find the system's Lagrangian and equations of motion. (Note: As in Problem 12-22 of Thornton & Marion, you may want to explain or justify why you do not have to consider the gravitational potential energy of the block, when you adopt a generalized coordinate s for the block, defined as a displacement from s_0 along the inclined surface.)

(c) Determine the eigenfrequencies and describe the corresponding normal mode motions.

• (a) $s_0 = l + \frac{mg\sin\alpha}{\kappa}$



• (b) Let x be the horizontal coordinate of the bottom left corner of the wedge, then the block is at $(x+s_0\cos\alpha+s\cos\alpha,\ h-s_0\sin\alpha-s\sin\alpha)$. Thus, $T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left[(\dot{x}+\dot{s}\cos\alpha)^2+(\dot{s}\sin\alpha)^2\right]$ and $U=\frac{1}{2}\kappa s^2$. As a note, $U=\frac{1}{2}\kappa(s_0+s-l)^2+mg(h-s_0\sin\alpha-s\sin\alpha)$ gives the same equations of motion.

• (c) $\omega_1=0$ (pure translation), and $\omega_2=\sqrt{\frac{\kappa(1+\beta)}{m(1+\beta\sin^2\alpha)}}$ where $\beta=\frac{m}{M}$.

10. Consider a system of masses constrained to move on a frictionless, horizontal hoop. Massless springs connect the masses and wrap around the hoop. For simplicity, let us set the radius of the hoop to be 1. Assume small oscillations.

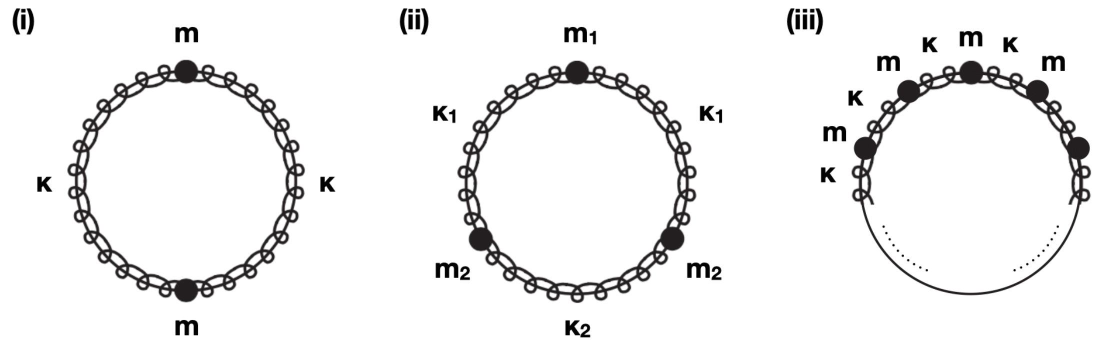
(a) First, consider the case of two identical masses m connected by springs with spring constant κ (see figure *(i)* below). Determine the eigenfrequencies and describe the normal mode motions.

(b) Now one mass is driven by an external sinusoidal force, $F_{\rm d}\cos\omega_{\rm d}t$. Find the particular solution for the motion of the masses. In particular, explain what happens if $\omega_{\rm d}=\sqrt{\frac{2\kappa}{m}}$.

(c) Let us now consider three masses connected by three springs (see figure *(ii)* below). In the following figure, the masses are shown at their equilibrium positions, located at 120° angular separations. First, for $m_1=m_2=m$ and $\kappa_1=\kappa_2=\kappa$, determine the eigenfrequencies and describe the normal mode motions.

(d) Assuming the same equilibrium positions as in (c), describe how your answer changes if $m_1=2m$, $m_2=m$, $\kappa_1=\kappa$, and $\kappa_2=2\kappa$. Then consider the case where m_1 is initially displaced 10° clockwise from its equilibrium position while the other two masses are held fixed. The three masses are then released from rest simultaneously. Find the resulting motion of each mass.

(e) Lastly, consider N identical masses connected by N identical springs (see figure *(iii)* below). Determine the eigenfrequencies and describe the normal mode motions. Discuss how your answer is different from and similar to the case of a loaded string discussed in Section 12.9.



• (a) $T = \frac{1}{2}m(\dot{\theta}_1^2 + \dot{\theta}_2^2)$, $U = \frac{1}{2}\kappa\left[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_1)^2\right]$. The normal modes are therefore, with $\omega_1 = 0$ (pure translation) and $\omega_2 = \sqrt{\frac{4\kappa}{m}}$,

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}(v_0 t + \theta_{i,0}), \quad \text{and} \quad \begin{pmatrix}1\\ -1\end{pmatrix} A_2 \cos(\omega_2 t - \delta_2).$$

• (b) By plugging a trial solution for the particular solution in the form of Eq.(3.55), $\theta_1 = B_1 \cos\omega_{\rm d} t$ and $\theta_2 = B_2 \cos\omega_{\rm d} t$, in the equations of motion, $m\ddot{x}_1 + 2\kappa(x_1 - x_2) = F_{\rm d}\cos\omega_{\rm d} t$ and $m\ddot{x}_2 + 2\kappa(x_2 - x_1) = 0$, two amplitudes B_1 and B_2 appear.

• (c) $T = \frac{1}{2}(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2 + m_2\dot{\theta}_3^2)$, $U = \frac{1}{2}\left[\kappa_1(\theta_1 - \theta_2)^2 + \kappa_1(\theta_1 - \theta_3)^2 + \kappa_2(\theta_2 - \theta_3)^2\right]$. But with $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$, the normal modes are therefore, with $\omega_2 = \omega_3 = \sqrt{\frac{3\kappa}{m}}$,

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}(v_0 t + \theta_{i,0}), \quad \begin{pmatrix}0\\ 1\\ -1\end{pmatrix} A_2 \cos(\omega_2 t - \delta_2), \quad \text{and} \quad \begin{pmatrix}1\\ 0\\ -1\end{pmatrix} A_3 \cos(\omega_3 t - \delta_3).$$

• (e) Section 12.9 and Problem 12-24 of Thornton & Marion tells us that the eigenfrequencies will be in the form of $\omega_r = \sqrt{\frac{4\kappa}{m}}\sin\frac{\gamma_r}{2}$ (similar to Eq.(12.144)), where γ_r needs to be determined by the boundary condition $a_{0r} = a_{Nr}$ (instead of $a_{0r} = a_{(N+1)r} = 0$ in Eq.(12.147)). The *periodic* boundary condition then quickly becomes $\cos(-\delta_r) = \cos(N\gamma_r - \delta_r)$, which yields $\gamma_r = \frac{2r\pi}{N}$ (compare with Eq.(12.149)). Note that here we observed the similar two-fold degeneracy of the eigenfrequencies found also in p.503 of Thornton & Marion.

11. A simple pendulum of mass m and length b is suspended from a cart of mass M. The cart moves horizontally along a frictionless rail. g is the gravitational acceleration. Assume small oscillations.

(a) First, find the system's Lagrangian and equations of motion (see figure *(i)* below). Determine the eigenfrequencies and describe the corresponding normal mode motions.

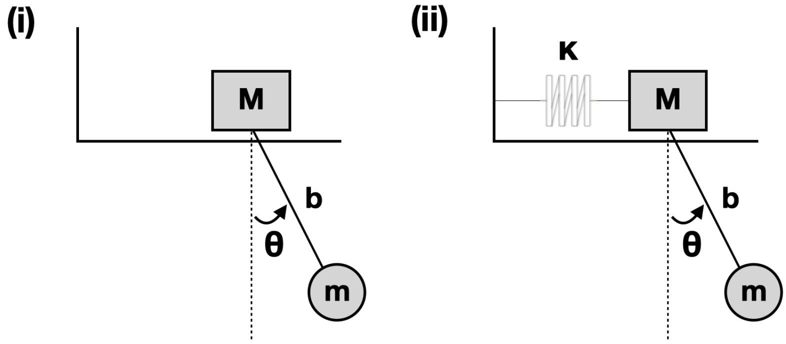
(b) Then, consider a case where a massless spring of spring constant κ is attached between the cart and an adjacent wall (see figure *(ii)* below). Determine the eigenfrequencies.

(c) Simplify the case in (b) by supposing $M = m$ and that the values of m, κ, and b are such that $\kappa b = 2mg$. Find the eigenfrequencies and describe the corresponding normal mode motions.

(d) Finally, let us carry out a quick sanity check on your answer in (b). Drop all the assumptions made in (c), but instead assume $m \ll M$ and $\frac{\kappa}{M} < \frac{g}{b}$ (i.e., loose spring). Find the eigenfrequencies and describe the corresponding normal mode motions. Explain their physical meanings.

• (a) From $T = \frac{1}{2}\left[M\dot{x}^2 + m(\dot{x}^2 + b^2\dot{\theta}^2 + 2\dot{x}b\dot{\theta}\cos\theta)\right] \simeq \frac{1}{2}\left[M\dot{x}^2 + m(\dot{x} + b\dot{\theta})^2\right]$ and $U = mgb(1 - \cos\theta) \simeq \frac{1}{2}mgb\theta^2$, one gets $\omega_1 = 0$ (pure translation) and $\omega_2 = \sqrt{\frac{g(M+m)}{bM}}$. Plugging ω_i back into the characteristic equation, the corresponding normal modes are

$$\begin{pmatrix}x\\ b\theta\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}(v_0 t + \theta_{i,0}), \quad \text{and} \quad \begin{pmatrix}1\\ -\frac{M+m}{m}\end{pmatrix} A_2 \cos(\omega_2 t - \delta_2).$$



• (b) From $T = \frac{1}{2}\left[M\dot{x}^2 + m(\dot{x} + b\dot{\theta})^2\right]$ and $U = \frac{1}{2}mgb\theta^2 + \frac{1}{2}\kappa x^2$, the eigenfrequencies are $\omega_1 = \left[\frac{g(M+m)+\kappa b}{2Mb} - \sqrt{\left(\frac{g(M+m)+\kappa b}{2Mb}\right)^2 - \frac{\kappa g}{Mb}}\right]^{\frac{1}{2}}$ and $\omega_2 = \left[\frac{g(M+m)+\kappa b}{2Mb} + \sqrt{\left(\frac{g(M+m)+\kappa b}{2Mb}\right)^2 - \frac{\kappa g}{Mb}}\right]^{\frac{1}{2}}$.

• (c) With $M = m$ and $\kappa b = 2mg$, the eigenfrequencies in (b) becomes $\omega_1 = \sqrt{\frac{(2-\sqrt{2})\kappa}{2m}}$ and $\omega_2 = \sqrt{\frac{(2+\sqrt{2})\kappa}{2m}}$. Plugging ω_i back into the characteristic equation, the corresponding normal modes are

$$\begin{pmatrix} x \\ b\theta \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} A_1 \cos(\omega_1 t - \delta_1), \text{ and } \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} A_2 \cos(\omega_2 t - \delta_2).$$

• (d) With $m \ll M$, the eigenfrequencies in (b) becomes $\omega_1 = \left[\left(\frac{g}{2b} + \frac{\kappa}{2M}\right) - \sqrt{\left(\frac{g}{2b} - \frac{\kappa}{2M}\right)^2}\right]^{\frac{1}{2}} = \sqrt{\frac{\kappa}{M}}$ and $\omega_2 = \left[\left(\frac{g}{2b} + \frac{\kappa}{2M}\right) + \sqrt{\left(\frac{g}{2b} - \frac{\kappa}{2M}\right)^2}\right]^{\frac{1}{2}} = \sqrt{\frac{g}{b}}$. Plugging ω_i back into the characteristic equation, the corresponding normal modes are

$$\begin{pmatrix} x \\ b\theta \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix} A_1 \cos(\omega_1 t - \delta_1), \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} A_2 \cos(\omega_2 t - \delta_2),$$

which confirms that the coupling between the two oscillators is effectively non-existent.

12. Starting from generalized coordinates q_k defined at the beginning of Section 12.4 (with $k = 1, 2, ..., n$), follow the texts in Sections 12.4 to 12.6 and review the general procedure to solve a problem of coupled oscillations — most of which were discussed in the class, but some left for your exercise. In particular, prove explicitly *(i)* that Eq.(12.37) can be derived from Eq.(12.34), *(ii)* that the eigenvectors $\mathbf{a}_r$ found in the process form an orthogonal set, and *(iii)* that Eq.(12.64) can be derived directly from Eq.(12.63) and Lagrange's equation of motion.

• For *(ii)* and *(iii)*, see Section 12.5 and Example 12.2, respectively.