Classical Mechanics II (Fall 2020): Homework #1

Due Sep. 29, 2020

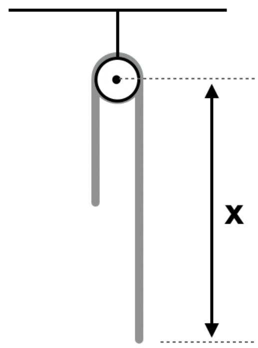
[0.5 pt each, total 6 pts]

1. Thornton & Marion, Problem 9-15

(Note: For Problem 9-15, discuss the difference between its setup and the one in Problem 9-21. Once you have the equation of motion, you may find it useful to assume that $\dot{x}^2$ can be written as $\dot{x}^2 = \sum_n a_n x^n$. This is a so-called power series solution — similar in philosophy to the Frobenius' power series method covered in e.g., Chapter 7.5 of Arfken, Weber & Harris, 7th ed., 2013.)

• $F = m\ddot{x} + \dot{m}\dot{x} = mg \;\rightarrow\; \ddot{x} + \frac{\dot{x}^2}{x} = g \;\rightarrow$ with $\dot{x}^2 = \sum_n a_n x^n$, you can write $\ddot{x} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{2}\frac{d(\dot{x}^2)}{dx} = \sum_n \frac{1}{2} n a_n x^{n-1}$ and $\frac{\dot{x}^2}{x} = \sum_n a_n x^{n-1}$. Equating the coefficients you get $\frac{1}{2} n a_n + a_n = g$ only if $n = 1$. Therefore, the only nonzero coefficient is $a_1 = \frac{2}{3} g$, which means $\dot{x}^2 = \frac{2}{3} g x$.

2. Thornton & Marion, Problem 9-20



• The energy conservation, $\left(\frac{m}{2}\right) ga = \frac{1}{2} m v^2$, yields $v = \sqrt{ga}$.

3. Thornton & Marion, Problem 9-43

• Following the notations in Figure 9-10, the conservation of momentum and energy means: *(i)* $mu_1 = mv_1 \cos\frac{\pi}{4} + 4mv_2 \cos\zeta$, *(ii)* $0 = mv_1 \sin\frac{\pi}{4} - 4mv_2 \sin\zeta$, and *(iii)* $\frac{1}{2} m u_1^2 \times \frac{5}{6} = \frac{1}{2} m v_1^2 + 2 m v_2^2$.

• This reduces to a set of two equations: *(i)'* $-u_1^2 = v_1^2 - 16v_2^2 - \sqrt{2}u_1v_1$ and *(ii)'* $5u_1^2 = 6v_1^2 + 24v_2^2$, from which we acquire $v_1 = \frac{\sqrt{2}+\sqrt{146/3}}{10}u_1 \simeq 0.84u_1$ and $\tan\zeta = \frac{4v_2\sin\zeta}{4v_2\cos\zeta} = \frac{v_1/\sqrt{2}}{u_1 - v_1/\sqrt{2}} \simeq 1.46$.

4. Thornton & Marion, Problem 9-45

• Use Eqs.(9.125) and (9.69) to eliminate ψ's but leave only θ's and x's.

5. Thornton & Marion, Problem 9-62

(Note: For Problem 9-62, note that we do not assume a constant burn rate of the fuel.)

• From Eq.(9.160) with $\dot{v} = 0$ and $g = \frac{1}{6}g_E$, one gets $-\frac{1}{6}mg_E = u\frac{dm}{dt} \;\rightarrow\; \int_0^t dt = -\frac{6u}{g_E}\int_{m_0}^{m}\frac{dm}{m}$.

6. Thornton & Marion, Problem 9-64

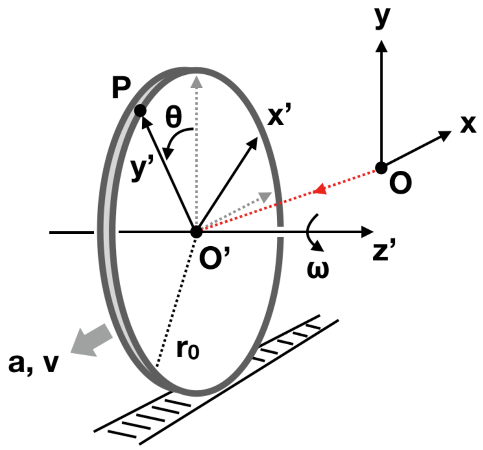
(Note: For Problem 9-64(b), use Eq.(2.21) with parameters given in Problem 9-63(b). For Problem 9-64(c), prove and use $g(y) = \frac{9.8}{(1+y/R_E)^2}\ \mathrm{m\,s^{-2}}$ where y is the altitude above Earth and R_E is Earth's radius.)

• (a) Note that we are trying to find the maximum height reached; the rocket keeps going up after the burnout. So for all subsequent problems, there always are two phases you need to consider: (1) the first phase with constant nonzero α, and (2) the second phase with $\alpha = 0$.

• (b) Expanding from Eqs.(9.161) and (9.162), $m\ddot{y} = u\alpha - mg - \frac{1}{2}c_{\mathrm{w}}\rho A\dot{y}^2$.

7. Thornton & Marion, Problem 10-2

(Note: For Problem 10-2, investigate the problem in two ways by using (a) an inertial reference frame $x - y$ centered on the initial position of the tire's center, and (b) a rotating noninertial reference frame $x' - y'$, as illustrated below.)

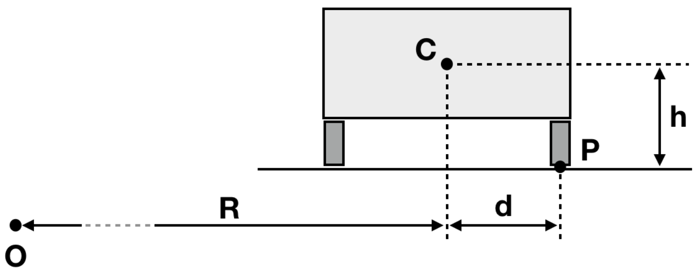


• (a) If we adopt the $x - y$ frame centered on the initial position of the tire's center, this question may remind some students of Problem 10 in Homework #1 of Classical Mechanics 1 (2020). The point P's velocity measured in the $x - y$ frame becomes $\mathbf{v}_{\mathrm{f},P} = (-v - r_0\dot{\theta}\cos\theta)\,\hat{\mathbf{e}}_x - r_0\dot{\theta}\sin\theta\,\hat{\mathbf{e}}_y$

with $r_0\dot{\theta} = v$. From this, one can show $|\mathbf{a}_{\mathrm{f},P}| = \left|\frac{d\mathbf{v}_{\mathrm{f},P}}{dt}\right| = a\sqrt{2 + 2\cos\theta + \frac{v^4}{a^2 r_0^2} - \frac{2v^2\sin\theta}{a r_0}}$ which peaks at $\theta = 2\pi - \tan^{-1}\left(\frac{v^2}{a r_0}\right)$.

• (b) Now, we adopt the $x' - y'$ frame rotating around the z'-axis aligned with the tire's axle. Using Eq.(10.23) with $\mathbf{v}_{\mathrm{r}} = \mathbf{a}_{\mathrm{r}} = 0$, $\boldsymbol{\omega} = \omega\hat{\mathbf{e}}'_z = \frac{v}{r_0}\hat{\mathbf{e}}'_z$, $\dot{\boldsymbol{\omega}} = \dot{\omega}\hat{\mathbf{e}}'_z = \frac{a}{r_0}\hat{\mathbf{e}}'_z$, $\ddot{\mathbf{R}}_{\mathrm{f}} = -a\cos\theta\,\hat{\mathbf{e}}'_x + a\sin\theta\,\hat{\mathbf{e}}'_y$, and $\mathbf{r} = r_0\,\hat{\mathbf{e}}'_y$ for the point P on the y'-axis, one gets $\mathbf{a}_{\mathrm{f},P} = -(a + a\cos\theta)\,\hat{\mathbf{e}}'_x + (a\sin\theta - \frac{v^2}{r_0})\hat{\mathbf{e}}'_y$. One can again show $|\mathbf{a}_{\mathrm{f},P}| = a\sqrt{2 + 2\cos\theta + \frac{v^4}{a^2 r_0^2} - \frac{2v^2\sin\theta}{a r_0}}$, the same as in (a).

8. A car of mass m travels with speed v on a horizontal, circular track of radius R. h is the height of the center of mass C above the ground, and $2d$ ($\ll R$) is the separation between the inner and outer wheels. The track is sufficiently rough that the wheels are not skidding. Show that the car will overturn if v is larger than $\sqrt{\frac{gRd}{h}}$. You are asked to consider the problem using (a) an inertial reference frame fixed on the ground and (b) a noninertial reference frame rotating at the same rate as the car.



• (a) When the car is about to overturn, no normal or frictional force acts on the inner wheel. So, the normal force acting on P (see the figure) is $N_P = mg$, and the frictional force on P becomes the centripetal force, $F_P = \frac{mv^2}{R}$. One last condition to add is that there is no net torque about C, meaning $F_P h = N_P d$. Combining the equations, one arrives at $\frac{mv^2 h}{R} = mgd$.

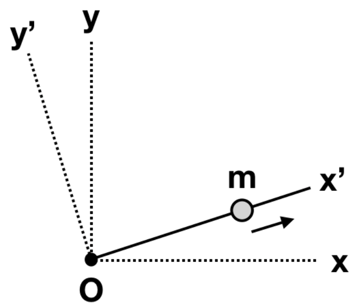
• (b) Again, when the car is about to overturn, no normal or frictional force acts on the inner wheel. So, the normal force acting on P is $N_P = mg$. Now the fictitious centrifugal force acting on C, $F_C = \frac{mv^2}{R}$, should be balanced by the frictional force on P, F_P, because the car is at rest in this frame. As there is no net torque about either C or P, one again reaches $\frac{mv^2 h}{R} = mgd$.

9. A rod of length b rotates with a constant angular speed ω about the z-axis through one end of the rod (point O) and perpendicular to the plane of rotation. A small bead of mass m, with a hole through it, is threaded on this frictionless rod.

(a) Write down the equation of motion of the bead in the $x' - y'$ frame rotating with the rod. Express the force that the rod exerts on the bead.

(b) The bead is now placed at O then pushed down the rod with an initial speed of $b\,\omega$ with respect to the rod. Calculate the time and velocity when the bead leaves the rod. In your answer you may choose to leave the trigonometric or hyperbolic functional form, or the inverse function thereof — e.g., $\sin(\square)$, $\sin^{-1}(\square)$, $\sinh(\square)$, $\sinh^{-1}(\square)$.

(c) The bead is now placed at the midpoint of the rod, and released from rest with respect to the rod. Calculate the time and velocity when the bead leaves the rod.



- (a) $\mathbf{F} = 2m\omega\dot{x}'\hat{\mathbf{e}}'_y$.

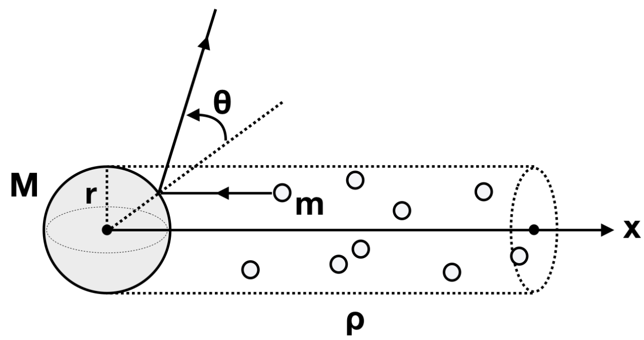
- (b) From Eq.(10.25), $m\ddot{x}' = m\omega^2 x'$. Then from Eq.(C.10) of Thornton & Marion, $x'(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$, which becomes $x'(t) = b\sinh(\omega t)$ considering the initial conditions $x'(0) = 0$ and $\dot{x}'(0) = b\,\omega$. Therefore, the bead leaves the end of the rod at $t_1 = \frac{1}{\omega}\sinh^{-1}(1)$.

- (c) $x'(t) = \frac{b}{2}\cosh(\omega t)$, $t_2 = \frac{1}{\omega}\cosh^{-1}(2)$

10. A spherical satellite of radius r and mass $M(t)$ is moving with velocity $v(t)$ through an atmosphere of uniform density ρ. The atmospheric particles are of identical sizes and masses, all initially at rest. Show that the retarding force on the satellite is written as $F_{\rm r} = -\rho(\pi r^2)v^2(t)$ in both of the following cases (resembling Eq.(2.21) in Thornton & Marion).

(a) Each atmospheric particle strikes the satellite and adheres to its surface.

(b) Each atmospheric particle strikes the satellite and bounces off from it elastically (see the figure below).



- (a) From momentum conservation, $Mv = (M + dM)(v + dv) \;\rightarrow\;$ thus, $F_{\rm r} = M\frac{dv}{dt} = -v\frac{dM}{dt}$ where $dM = \rho(\pi r^2)v dt$.

- (b-1) When a particle strikes the satellite's surface at angle θ measured from the x-axis (see the figure), the momentum conservation in the x direction gives $-mv = \Delta p_x + mv\cos 2\theta$. Note that we now work in the frame where the satellite is at rest, and we don't have to worry about the momentum transferred in other directions perpendicular to the x-axis (why?).

- (b-2) Meanwhile, the number of particles hitting the surface at $[\theta, \theta + d\theta]$ in dt is $dN = \left(\frac{\rho}{m}\right) 2\pi(r\sin\theta)d(r\sin\theta)v dt$.

- (b-3) Therefore, $F_{\rm r} = \frac{\int \Delta p_x dN}{dt} = \int_0^{\frac{\pi}{2}} (-mv - mv\cos 2\theta) \cdot \left(\frac{\rho}{m}\right) 2\pi(r\sin\theta)d(r\sin\theta)v$.

11. Work out Example 9.2. In particular, prove explicitly the statements made in the last paragraph (the bottom of p.335) about the continuity — or discontinuity — of the tension on either side of the bottom bend, for both free fall and energy-conserving cases.

- For the free fall case, $T_1 = T_{\text{Eq.(9.16)}} - \rho\left(\frac{b+x}{2}\right)g = \rho g x = \frac{1}{2}\rho\dot{x}^2$.

- For the energy-conserving case, $T_1 = T_{\text{Eq.(9.18)}} - \rho\left(\frac{b+x}{2}\right)g = \frac{\rho g}{4}\cdot\frac{2bx-x^2}{b-x} = \frac{1}{4}\rho\dot{x}^2$.

12. In the class we discussed the kinematics of elastic collisions. Starting from the initial energy of the system in the LAB and CM systems, Eqs.(9.78) and (9.79), follow step by step the logical procedure that leads to Eq.(9.87a), the LAB energy of the particle m_1 written with the CM scattering angle θ. Continue to derive Eqs.(9.88), (9.90), (9.91) and (9.92) — which were briefly discussed in the class but left for your exercise. An ambitious student seeking an additional +0.5 point may venture to derive Eq.(9.87b).

- From Eqs.(9.74) and (9.87a), $\frac{T_2}{T_0} = 1 - \frac{T_1}{T_0} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) = \frac{2m_1m_2}{(m_1+m_2)^2}(1+\cos 2\zeta) = \frac{4m_1m_2}{(m_1+m_2)^2}\cos^2\zeta$.