Classical Mechanics II (Fall 2020): Final Exam Solution

Dec. 12, 2020

[total 25 pts, closed book/cellphone, no calculator, 90 minutes]

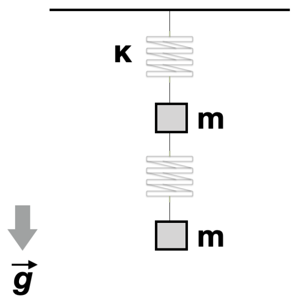
1. [4 pt] Consider a system of two identical masses of negligible sizes and mass m each which are connected to the ceiling by two massless springs of spring constant κ and original unstretched length l each, as shown in the figure below. The masses are constrained to move only vertically in the uniform gravitational field g.

(a) [1 pt] When both masses are at rest, find their equilibrium positions from the ceiling.

(b) [1 pt] Assuming small oscillations, find the system's Lagrangian and equations of motion. (Note: You may want to explain or justify why you do not have to consider the gravitational potential energies of the masses, when you adopt a coordinate for each mass measured from its equilibrium position.)

(c) [1 pt] Determine the eigenfrequencies and describe the corresponding normal mode motions.

(d) [1 pt] Find an example initial condition for the system that later makes the two masses oscillate in the *symmetrical* normal mode.



• (a) Distances from the ceiling at equilibrium: upper mass $x_{1,0} = l + \frac{2mg}{\kappa}$, lower mass $x_{1,0} = 2l + \frac{3mg}{\kappa}$.

• (b) Let x_1 and x_2 be the displacements from $x_{1,0}$ and $x_{2,0}$, respectively. Then, $T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$ and $U = \frac{1}{2}\kappa x_1^2 + \frac{1}{2}\kappa(x_2 - x_1)^2$. The equations of motion becomes $m\ddot{x}_1 + 2\kappa x_1 - \kappa x_2 = 0$ and $m\ddot{x}_2 + \kappa x_2 - \kappa x_1 = 0$. The constant downward gravitational force only shifts the equilibrium positions but does not affect the normal mode motions.

• (c) $\omega_1 = \sqrt{\frac{(3-\sqrt{5})\kappa}{2m}}$, and $\omega_2 = \sqrt{\frac{(3+\sqrt{5})\kappa}{2m}}$. Plugging ω_i back into the characteristic equation, the corresponding normal modes are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} A_1 \cos(\omega_1 t - \delta_1), \quad \text{and} \quad \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} A_2 \cos(\omega_2 t - \delta_2).$$

• (d) To suppress the antisymmetrical mode and leave only the symmetrical mode, one possible way to initialize the system is to set: $x_2(0) = \frac{1+\sqrt{5}}{2} x_1(0)$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$.

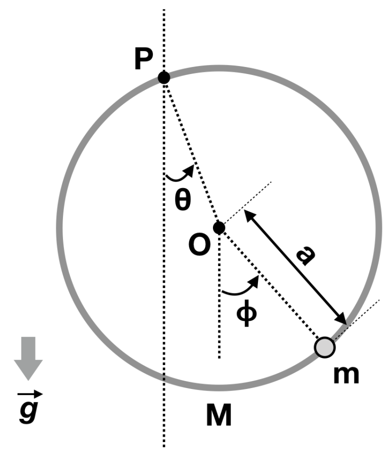
2. [5 pt] A bead of mass m is threaded on a frictionless, circular wire hoop of mass M and radius a. The hoop is pivoted on its rim at point P (see the figure) so it can swing freely in its plane in the presence of the uniform gravitational field g.

(a) [1 pt] Determine the Lagrangian of the system using the generalized coordinates θ and ϕ shown in the figure.

(b) [1 pt] Now assuming small oscillations about the equilibrium points $\theta = \phi = 0$, find Lagrange's equations of motion of the system.

(c) [2 pt] Determine the eigenfrequencies and describe the corresponding normal mode motions.

(d) [1 pt] Finally let us carry out a quick sanity check on your answer in (c). When $m \ll M$, the system becomes a simpler physical pendulum of mass M. Assuming small oscillations about the equilibrium point $\theta = 0$, find $\theta(t)$ and its oscillation frequency. Verify that, in the limit $m \ll M$, the displacement $\theta(t)$ found in (c) reduces to what you acquired here in (d).



• (a) $T = \frac{1}{2}a^2 \left[(2M+m)\dot{\theta}^2 + m\dot{\phi}^2 + 2m\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right]$, $U = ga\left[(M+m)(1-\cos\theta) + m(1-\cos\phi)\right]$.

• (b) $T = \frac{1}{2}a^2 \left[(2M+m)\dot{\theta}^2 + m\dot{\phi}^2 + 2m\dot{\theta}\dot{\phi}\right]$, $U = \frac{1}{2}ga\left[(M+m)\theta^2 + m\phi^2\right]$.

• (c) From the Lagrange's equations of motion, one gets eigenfrequencies $\omega_1 = \sqrt{\frac{g}{2a}}$ (in-phase) and $\omega_2 = \sqrt{\frac{(1+\alpha)g}{a}}$ (out of phase) where $\alpha = \frac{m}{M}$. Plugging ω_i back into the characteristic equation, the corresponding normal modes are

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 \cos(\omega_1 t - \delta_1), \quad \text{and} \quad \begin{pmatrix} -\alpha \\ 1+\alpha \end{pmatrix} A_2 \cos(\omega_2 t - \delta_2).$$

3. [5 pt] A string of length L is fixed at both ends, and is initially given no displacement but set into motion by being struck over a length $2s$ about its center. The center section is given an initial velocity v_0 at $t = 0$.

(a) [2 pt] Let us attempt to describe the subsequent motion of the string in terms of normal modes. For this, you may want to find the displacement $\Psi(x,t)$ as a function of position x and time t, and the characteristic frequencies ω_r (for the rth normal mode). For our trial solution $\Psi(x,t) = \sum_r \beta_r \psi_r(x)\chi_r(t)$, propose your choices for $\psi_r(x)$ and $\chi_r(t)$, and explain your reasoning.

(b) [2 pt] Acquire the exact form of $\Psi(x,t)$ and ω_r for the string described above.

(c) [1 pt] Among the "harmonics" of characteristic frequencies, which are missing when describing the vibration here? Briefly explain why.

- (a) $\sum_r \beta_r e^{i\omega_r t} \sin\left(\frac{r\pi x}{L}\right)$, or its real part $\sum_r (\mu_r \cos\omega_r t - \nu_r \sin\omega_r t)\sin\left(\frac{r\pi x}{L}\right)$.
- (b) From Eqs.(13.5) to (13.8), $\omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}}$, $\beta_r = \mu_r + i\nu_r$, $\mu_r = 0$, but,

$$\begin{aligned}
\nu_r &= -\frac{2}{\omega_r L}\int_{\frac{L}{2}-s}^{\frac{L}{2}+s} v_0 \sin\left(\frac{r\pi x}{L}\right) dx = \frac{2v_0}{r\pi\omega_r}\left[\cos\frac{r\pi(L/2+s)}{L} - \cos\frac{r\pi(L/2-s)}{L}\right] \\
&= -\frac{4v_0}{r\pi\omega_r}\sin\frac{r\pi}{2}\sin\frac{r\pi s}{L}
\end{aligned}$$

- (c) One can observe that $\nu_r = -\frac{4v_0}{r^2\pi\omega_1}(-1)^{\frac{r-1}{2}}\sin\frac{r\pi s}{L}$ when r is odd, but vanishes when r is even.

4. [4 pt] The Lorentz transformation matrix between two inertial reference frames K and K' which move along their x_1- and x_1'-axes with a uniform relative velocity v, can be written as

$$\boldsymbol{\lambda} = \lambda_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ with indices μ, $\nu = 1, 2, 3, 4$. The quantity $\mathbb{X} = x_\mu = (\mathbf{x}, ict)$ is a four-vector as each of its components transforms according to the relation $\mathbb{X}' = x'_\mu = \boldsymbol{\lambda}\mathbb{X}$, with a 3-dimensional position vector $\mathbf{x}$ and time t.

(a) [1 pt] Prove that one can write a four-vector momentum of a particle (of mass m, 3-dimensional relativistic momentum $\mathbf{p}$, relativistic total energy E) as $\mathbb{P} = (\mathbf{p}, i\frac{E}{c})$. (Note: You will have to explicitly demonstrate that $\mathbb{P}$ is a four-vector. You may first want to show that $\mathbb{V} = \frac{d\mathbb{X}}{d\tau} = (\gamma\mathbf{u}, i\gamma c)$ is a four-vector that transforms according to the same $\boldsymbol{\lambda}$ above. Here τ is the Lorentz invariant proper time, and $\mathbf{u}$ is the 3-dimensional velocity vector of the particle.)

(b) [1 pt] By first showing $\mathbb{V}^2 = -c^2$, prove $\mathbb{P}^2 = -m^2c^2$. Show also that this leads to a familiar equation, $E^2 = p^2c^2 + m^2c^4$, where $p^2 = \mathbf{p}^2$.

(c) [2 pt] Show that the equation

$$\nabla^2\Psi(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{x},t)}{\partial t^2} = 0$$

is invariant under the Lorentz transformation. What is the physical meaning you can draw from this observation? (Note: You may want to simplify the math by demonstrating that the above equation is equal to

$$\sum_{\mu=1}^{4}\frac{\partial^2\Psi(x_\mu)}{\partial x_\mu^2} = 0,$$

and then proceed to show $\sum\limits_\mu \frac{\partial^2}{\partial x_\mu^2} = \sum\limits_\mu \frac{\partial^2}{\partial x_\mu'^2}$.)

- (a) Eqs.(14.83)-(14.91) of Thornton & Marion.
- (b) Eqs.(14.93)-(14.95) of Thornton & Marion.
- (c-1) From $x'_\mu = \sum\limits_\nu \lambda_{\mu\nu}x_\nu$, one can acquire $\frac{\partial}{\partial x'_\mu} = \sum\limits_\nu \lambda_{\mu\nu}\frac{\partial}{\partial x_\nu}$, then $\frac{\partial^2}{\partial x_\mu'^2} = \sum\limits_\nu\sum\limits_\sigma \lambda_{\mu\nu}\lambda_{\mu\sigma}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\sigma}$. We also know $\sum\limits_\mu \lambda_{\mu\nu}\lambda_{\mu\sigma} = \delta_{\nu\sigma}$ by transforming x_μ to x'_μ then back to x_μ — i.e., $x_\mu = \sum\limits_\sigma \lambda_{\mu\sigma}x'_\sigma = \sum\limits_\sigma \lambda_{\mu\sigma}\left(\sum\limits_\nu \lambda^t_{\sigma\nu}x_\nu\right) = \sum\limits_\sigma \lambda_{\mu\sigma}\left(\sum\limits_\nu \lambda_{\nu\sigma}x_\nu\right) = \sum\limits_\nu\left(\sum\limits_\sigma \lambda_{\mu\sigma}\lambda_{\nu\sigma}\right)x_\nu$.
- (c-2) Hence, $\sum\limits_\mu \frac{\partial^2}{\partial x_\mu'^2} = \sum\limits_\mu\sum\limits_\nu\sum\limits_\sigma \lambda_{\mu\nu}\lambda_{\mu\sigma}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\sigma} = \sum\limits_\nu\sum\limits_\sigma\left(\sum\limits_\mu \lambda_{\mu\nu}\lambda_{\mu\sigma}\right)\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\sigma} = \sum\limits_\nu\sum\limits_\sigma \delta_{\nu\sigma}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\sigma} = \sum\limits_\nu \frac{\partial^2}{\partial x_\nu^2} = \sum\limits_\mu \frac{\partial^2}{\partial x_\mu^2}$. The wave equation describing the propagation of electromagnetic waves or light waves in free space is, as expected, shown to be Lorentz invariant. Of course, this equation is not Galilean invariant.

5. [4 pt] A particle (particle 1) of mass m_1 and relativistic total energy E_1 decays in flight into two identical particles (particle 2 and 3) of mass m_2. (An example would be a neutral pion π^0 decaying into two photons.) θ_2 and θ_3 are the LAB angles particle 2 and 3 emerges at, respectively, relative to the particle 1's direction. Consider two scenarios below.

(a) [2 pt] In the first scenario, the two created particles emerge on each side of the particle 1's direction with equal angles $\theta_2 = \theta_3 = \theta$. Find $\cos\theta$ as a function of m_1, m_2 and E_1.

(b) [2 pt] In the second scenario, particle 2 is emitted at $\theta_2 = 90°$. Find the energies of the created particles (E_2 and E_3 for particle 2 and 3, respectively) as functions of m_1, m_2 and E_1. (Note: You may expedite your calculation by using the conservation of four-vector momenta, $\mathbb{P}_1 = \mathbb{P}_2 + \mathbb{P}_3$, while noticing $\mathbb{P}^2 = -m^2c^2$ derived in Problem 4.(b) above.)

- (a-1) Generally speaking, in the LAB frame, the four-vector momentum of particle 1 is $\mathbb{P}_1 = \left(p_1, 0, 0, i\frac{E_1}{c}\right)$ with $E_1^2 = p_1^2c^2 + m_1^2c^4$, and those of particle 2 and 3 are $\mathbb{P}_2 = (p_2\cos\theta_2, p_2\sin\theta_2, 0, i\frac{E_2}{c})$ and $\mathbb{P}_3 = (p_3\cos\theta_3, -p_3\sin\theta_3, 0, i\frac{E_3}{c})$, respectively, with $E_2^2 = p_2^2c^2 + m_2^2c^4$ and $E_3^2 = p_3^2c^2 + m_2^2c^4$.

• (a-2) Therefore, from energy and momentum conservation $\mathbb{P}_1 = \mathbb{P}_2 + \mathbb{P}_3$ with $\theta_2 = \theta_3 = \theta$, one gets $E_1 = 2E_2$ and $p_1 = 2p_2 \cos\theta$. $\rightarrow$ Hence, $E_2 = \frac{E_1}{2}$ and $\cos\theta = \frac{p_1}{2p_2} = \frac{\sqrt{E_1^2 - m_1^2 c^4}}{2\sqrt{(E_1/2)^2 - m_2^2 c^4}}$.

• (b-1) By squaring $\mathbb{P}_1 - \mathbb{P}_2 = \mathbb{P}_3$ and using $\mathbb{P}_1^2 = -m_1^2 c^2$ and $\mathbb{P}_2^2 = \mathbb{P}_3^2 = -m_2^2 c^2$, one gets $\mathbb{P}_1^2 - 2\mathbb{P}_1\mathbb{P}_2 + \mathbb{P}_2^2 = -m_1^2 c^2 + \frac{2E_1E_2}{c^2} - m_2^2 c^2 = \mathbb{P}_3^2 = -m_2^2 c^2$. Here we used $\mathbb{P}_1\mathbb{P}_2 = -\frac{E_1E_2}{c^2}$ with $\theta_2 = \frac{\pi}{2}$. $\rightarrow$ Therefore one acquires $E_2 = \frac{m_1^2 c^4}{2E_1}$. $E_1 = E_2 + E_3$ then yields $E_3 = E_1 - \frac{m_1^2 c^4}{2E_1}$.

• (b-2) Alternatively, from $\mathbb{P}_1 = \mathbb{P}_2 + \mathbb{P}_3$ with $\theta_2 = \frac{\pi}{2}$, one has $E_1 = E_2 + E_3$, $p_1 = p_3 \cos\theta_3$, and $p_2 = p_3 \sin\theta_3 = p_1 \tan\theta_3$. $\rightarrow$ From this, $E_1 = \sqrt{p_2^2 c^2 + m_2^2 c^4} + \sqrt{p_3^2 c^2 + m_2^2 c^4} = \sqrt{p_1^2 c^2 \tan^2\theta + m_2^2 c^4} + \sqrt{p_1^2 c^2 \sec^2\theta + m_2^2 c^4}$. Noticing $\sec^2\theta = 1 + \tan^2\theta$, the preceding equation eventually yields $E_2 = \frac{m_1^2 c^4}{2E_1}$.

6. [3 pt] Throughout the semester we discussed many examples in which concepts in classical mechanics are utilized in explaining daily phenomena. We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques like order-of-magnitude estimation and/or dimensional analysis. For the two questions here, you have been given a fair amount of heads-up during the classes.

(a) [2 pt] In the last class of the semester, five of your peers presented their term projects. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 2-3 sentences is expected to clearly convey the core physics idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem set-up. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in ∼3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

• (a) See the student presentation slides for Lecture 15-1 that include the collection of term project presentations by five students on December 8.

• (b) See the class slides for Lecture 14-2 that include many example problems, and the grading guideline.