Classical Mechanics II (Fall 2020): Midterm Exam Solution

Oct. 24, 2020

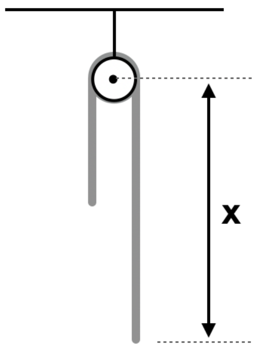
[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. [3 pt] A very flexible rope of mass m, uniform density ρ and total length $2b$ hangs in equilibrium over a frictionless horizontal pin of negligible diameter. The rope is given a slight vertical displacement causing it to slowly roll off the pin. The rope is prevented from lifting off the pin, and only gravity acts on the rope. The length x of the longer portion of the rope will thus grow in time. The pin is positioned higher than $2b$ from the ground.

(a) [1 pt] Using energy conservation, show that the rope's velocity is written as $\dot{x} = \sqrt{\frac{g}{b}}(x - b)$ where g is the gravitational acceleration. Then find the rope's acceleration $\ddot{x}$ as a function of x.

(b) [1 pt] Find the rope's acceleration, this time using the Lagrangian method.

(c) [1 pt] Find the tension in the rope, F_T, at the top position right next to the pin.



• (a) $T(\dot{x}) = \frac{1}{2}m\dot{x}^2,\ U(x) = -\rho x \cdot g \cdot \left(\frac{x}{2}\right) - \rho(2b - x) \cdot g \cdot \frac{2b-x}{2} \ \rightarrow\ T(\dot{x}) + U(x) = T(0) + U(b)$ yields $\dot{x}^2 = \frac{g}{b}(x - b)^2$. Differentiating the equation gives $\ddot{x} = \frac{g}{b}(x - b)$.

• (b) The Lagrangian equation of motion gives $\rho g x - \rho g(2b - x) - m\ddot{x} = 0$, where $m = 2\rho b$.

• (c-1) Following the similar procedure as in Eq.(9.147), one can observe that the change in momentum of the longer portion of the rope is $p(t+dt) - p(t) = (m+dm)(v+dv) - mv - vdm = mdv$. Note that the small mass element added to the longer portion in dt has been moving at speed $v\ (= \dot{x})$ already (i.e., no relative velocity between m and dm).

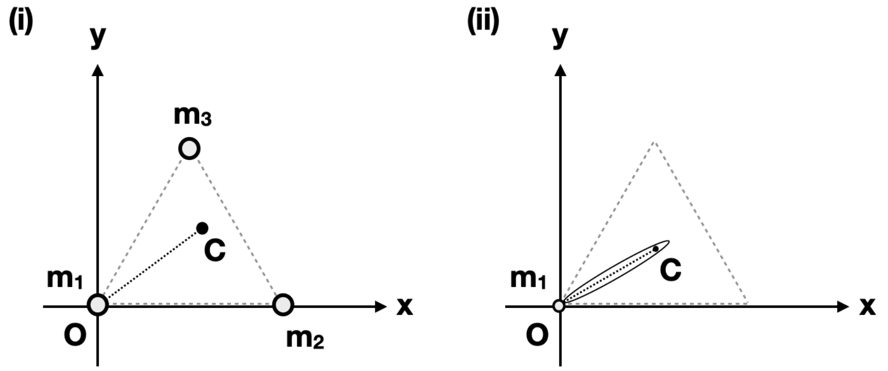
• (c-2) Therefore, the tension F_T at the top right next to the pin satisfies $m\frac{dv}{dt} = \rho x \cdot \ddot{x} = \rho x \cdot g - F_T$, or, $\rho(2b - x) \cdot \ddot{x} = F_T - \rho(2b - x) \cdot g$. Plugging $\ddot{x} = \frac{g}{b}(x - b)$ into the preceding equation, one acquires $F_T = \frac{\rho g}{b}x(2b - x)$. A quick sanity check shows $F_T(x = 2b) = 0$.

2. [4 pt] Consider a system of three stars with masses m_1, m_2 and m_3, placed at the corners of an equilateral triangle of side L (see figure (i) below). The total mass of the system is $M = m_1 + m_2 + m_3$. Only gravity acts between the bodies.

(a) [1 pt] First, assume that the stars are in a rotational motion under mutual gravitational attraction while keeping the relative separation of each pair of stars unchanged on the original triangle's plane. Using the coordinate seen below, find the location of the center of mass C.

(b) [1 pt] By demonstrating that the combined force acting on each star is directed toward C, show that each star is in a circular motion about C. Show that the period of the circular motions of all three stars is $T_{\rm circ} = 2\pi\sqrt{\frac{L^3}{GM}}$.

(c) [2 pt] Now assume that the stars are of equal masses, and initially at rest. Show that the stars will collapse to the center of the triangle C with no tangential velocity, and collide with one another in time $T_{\rm col} = \frac{1}{4\sqrt{2}}T_{\rm circ} = \pi\sqrt{\frac{L^3}{8GM}}$. (Note: You may want to use the observation that a star collapsing from O to C traces a *severely* flattened elliptical orbit of a semi-major axis $\frac{\overline{OC}}{2}$. See figure (ii) below.)



• (a) The location of C measured from m_1 is $\mathbf{R}_1 = \frac{L}{M}\left[\left(m_2 + \frac{m_3}{2}\right)\hat{\mathbf{e}}_x + \frac{\sqrt{3}m_3}{2}\hat{\mathbf{e}}_y\right]$.

• (b) The combined force acting on m_1 is $\mathbf{F}_1 = \frac{Gm_1}{L^2}\left[\left(m_2 + \frac{m_3}{2}\right)\hat{\mathbf{e}}_x + \frac{\sqrt{3}m_3}{2}\hat{\mathbf{e}}_y\right] = \frac{GMm_1}{L^3}\mathbf{R}_1$. Thus, m_1 is shown to be acted upon by a central force with its center at C, which is true for all m_i's by permutation of indices. Finally, $T_{\rm circ} = \frac{2\pi|\mathbf{R}_1|}{v_1} = \frac{2\pi|\mathbf{R}_1|}{\sqrt{|\mathbf{F}_1||\mathbf{R}_1|/m_1}} = 2\pi\sqrt{\frac{m_1|\mathbf{R}_1|}{|\mathbf{F}_1|}} = 2\pi\sqrt{\frac{L^3}{GM}}$.

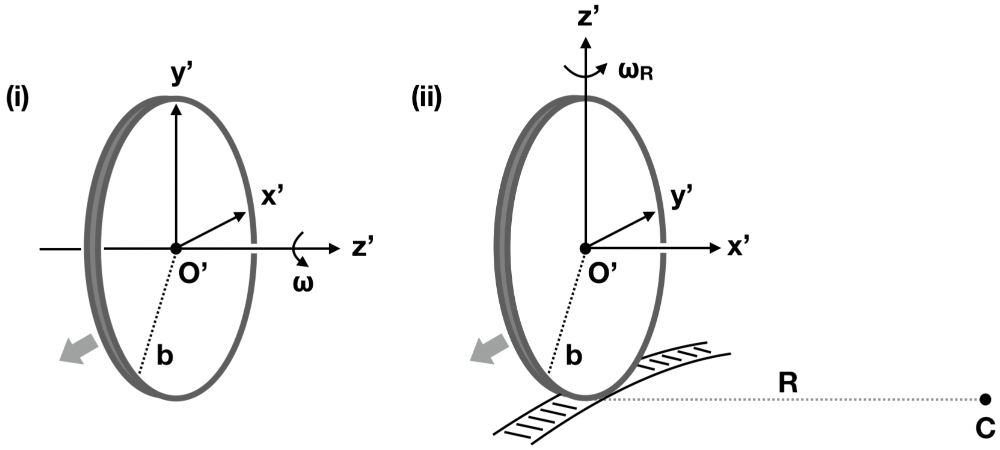
• (c) Because m_1 is acted upon by a combined force toward C but has no initial velocity, m_1 will simply collapse to C with no tangential velocity. The semi-major axis in the collapsing "orbit" in figure (ii) (i.e., $\frac{\overline{OC}}{2}$) is a half of that in the circular orbit found in (b). And, the collapsing time $T_{\rm col}$ is a half of the orbital period of this collapsing "orbit." Therefore, Kepler's 3rd law leads us to $\left(\frac{2T_{\rm col}}{T_{\rm circ}}\right)^2 = \left(\frac{\overline{OC}/2}{\overline{OC}}\right)^3 \rightarrow T_{\rm col} = \frac{1}{2}T_{\rm circ}\left(\frac{\overline{OC}/2}{\overline{OC}}\right)^{3/2} = \frac{1}{4\sqrt{2}}T_{\rm circ}$.

3. [4 pt] A car with wheels of radius b travels in the following ways. Find the acceleration, relative to the ground, of the highest point on one of its wheels. Depending on the problem setup, you may consider different types of rotating coordinate systems illustrated below (see next page).

(a) [1 pt] With constant forward velocity $\mathbf{v}_0$.

(b) [2 pt] With constant forward acceleration $\mathbf{a}_0$ and instantaneous velocity $\mathbf{v}$.

(c) [1 pt] With constant forward speed v_0 around a circular track of radius R.



• (a) We adopt the $x'-y'$ frame rotating around the z'-axis aligned with the wheel's axle. Using Eq.(10.23) with $\ddot{\mathbf{R}}_\mathrm{f} = \mathbf{v}_\mathrm{r} = \mathbf{a}_\mathrm{r} = 0$, $\boldsymbol{\omega} = \omega\hat{\mathbf{e}}'_z = \frac{v_0}{b}\hat{\mathbf{e}}'_z$, $\dot{\boldsymbol{\omega}} = 0$, and $\mathbf{r} = b\,\hat{\mathbf{e}}'_y$ (assuming $\hat{\mathbf{e}}'_y$ is the upward vertical at this instant), you get $\mathbf{a}_\mathrm{f} = -\omega^2 b\,\hat{\mathbf{e}}'_y = -\frac{v_0^2}{b}\hat{\mathbf{e}}'_y$.

• (b) Now with $\mathbf{v}_\mathrm{r} = \mathbf{a}_\mathrm{r} = 0$, $\boldsymbol{\omega} = \omega\hat{\mathbf{e}}'_z = \frac{v}{b}\hat{\mathbf{e}}'_z$, $\dot{\boldsymbol{\omega}} = \dot{\omega}\hat{\mathbf{e}}'_z = \frac{a_0}{b}\hat{\mathbf{e}}'_z$, $\mathbf{r} = b\,\hat{\mathbf{e}}'_y$, and $\ddot{\mathbf{R}}_\mathrm{f} = -a_0\,\hat{\mathbf{e}}'_x$ (assuming $-\hat{\mathbf{e}}'_x$ is the forward horizontal at this instant), you get $\mathbf{a}_\mathrm{f} = -a_0\,\hat{\mathbf{e}}'_x - \dot{\omega} b\,\hat{\mathbf{e}}'_x - \frac{v^2}{b}\hat{\mathbf{e}}'_y = -2a_0\,\hat{\mathbf{e}}'_x - \frac{v^2}{b}\hat{\mathbf{e}}'_y$.

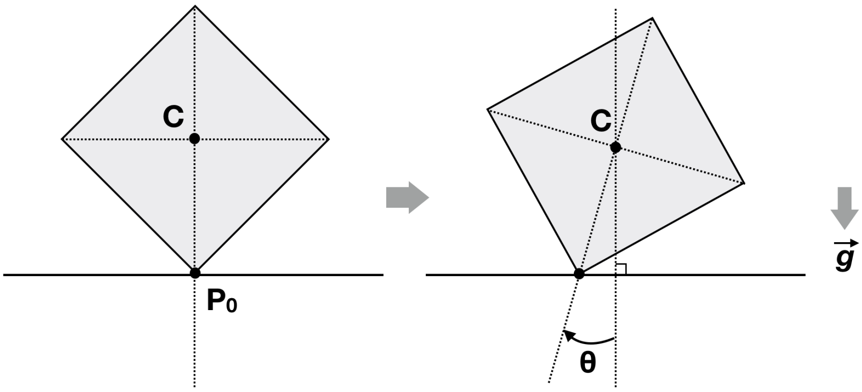
• (c) Now we adopt a different $x'-y'$ frame rotating (likely much more slowly) around the upward vertical z'-axis through the center of the wheel as the car travels in the track. Here, $\boldsymbol{\omega} = \omega_R\hat{\mathbf{e}}'_z = \frac{v_0}{R}\hat{\mathbf{e}}'_z$, $\dot{\boldsymbol{\omega}} = 0$, $\mathbf{r} = b\,\hat{\mathbf{e}}'_z$, $\ddot{\mathbf{R}}_\mathrm{f} = \frac{v_0^2}{R}\,\hat{\mathbf{e}}'_x$ (assuming $\hat{\mathbf{e}}'_x$ is aligned with the wheel's axle and thus towards the center of the track), $\mathbf{v}_\mathrm{r} = -v_0\hat{\mathbf{e}}'_y$, and $\mathbf{a}_\mathrm{r} = -\frac{v_0^2}{b}\,\hat{\mathbf{e}}'_z$, you get $\mathbf{a}_\mathrm{f} = \frac{v_0^2}{R}\,\hat{\mathbf{e}}'_x - \frac{v_0^2}{b}\,\hat{\mathbf{e}}'_z + \frac{2v_0^2}{R}\,\hat{\mathbf{e}}'_x = \frac{3v_0^2}{R}\,\hat{\mathbf{e}}'_x - \frac{v_0^2}{b}\,\hat{\mathbf{e}}'_z$.

4. [4 pt] A homogenous cube, each edge of which has a length b, is initially in an unstable equilibrium with its one edge in contact with a horizontal plane. The cube is then given a slight nudge and allowed to fall in a uniform gravitational field g. P_0 is the point fixed on the ground that is right below the cube's center of mass, C, in the edge initially in contact with the plane.

(a) [1 pt] First, assume that the edge cannot slide on the plane. Find the angular acceleration of the cube's rotation as a function of θ (see the figure). You may first want to find the location of C with respect to P_0.

(b) [1 pt] Determine the angular velocity of the cube's rotation when one of its faces strikes the plane.

(c) [2 pt] Now assume that the plane is frictionless (i.e., sliding can occur without friction). Determine the angular velocity of the cube's rotation when one of its faces strikes the plane.



- (a) In the $x-y$ coordinate with origin P_0, C is located at $(\frac{b}{\sqrt{2}}\sin\theta, \frac{b}{\sqrt{2}}\cos\theta)$. From this, $T = T_{\rm CM} + T_{\rm rot} = \frac{1}{2}M\left(\frac{b^2\dot\theta^2}{2}\right) + \frac{1}{12}Mb^2\dot\theta^2$ and $U = Mg\cdot\frac{b}{\sqrt{2}}\cos\theta$. And the Lagrange's equation of motion gives $\frac{2}{3}Mb^2\ddot\theta = \frac{1}{\sqrt{2}}Mgb\sin\theta \;\rightarrow\; \ddot\theta = \frac{3\sqrt{2}g}{4b}\sin\theta$.

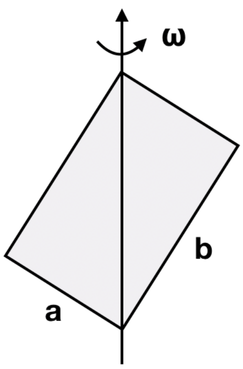
- (b) Multiplying both sides of the above equation by $\dot\theta$ and integrating, one gets $\dot\theta^2_{\theta=\pi/4} = \frac{3\sqrt{2}g}{2b}\int_0^{\pi/4}\sin\theta d\theta = \frac{3g}{2b}(\sqrt{2}-1)$. This can also be acquired using energy conservation, $\frac{1}{3}Mb^2\dot\theta^2_{\theta=\pi/4} = Mgb(\frac{1}{\sqrt{2}}-\frac{1}{2})$.

- (c) C is now located at $R(0, \frac{b}{\sqrt{2}}\cos\theta)$, always right above P_0, as the gravity causes C to move only vertically. Thus, $T = T_{\rm CM} + T_{\rm rot} = \frac{1}{2}M\left(\frac{b^2\dot\theta^2\cos^2\theta}{2}\right) + \frac{1}{12}Mb^2\dot\theta^2$ and $U = Mg\cdot\frac{b}{\sqrt{2}}\cos\theta$. Using energy conservation, $\left(\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{12}\right)Mb^2\dot\theta^2_{\theta=\pi/4} = Mgb(\frac{1}{\sqrt{2}}-\frac{1}{2}) \;\rightarrow\; \dot\theta^2_{\theta=\pi/4} = \frac{12g}{5b}(\sqrt{2}-1)$.

5. [3 pt] A thin rectangular plate with mass m and sides a and b is spinning around its diagonal with constant angular velocity ω.

(a) [1 pt] After defining your body coordinate system, find the angular momentum of the plate.

(b) [2 pt] Find the magnitude and direction of the torque $\mathbf{N}$ that needs to be applied to keep the plate spinning with constant ω. Describe what happens when $a = b$.



• (a) We choose a principal axes body coordinate system with its origin O at the center of mass of the plate, and with $\hat{\mathbf{e}}_1$ parallel to edge b, $\hat{\mathbf{e}}_2$ parallel to edge a, and $\hat{\mathbf{e}}_3$ according to the right-hand rule. Then,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \frac{m}{12}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \omega \begin{pmatrix} \frac{b}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix} = \frac{mab\omega}{12\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

• (b) From Eq.(11.117) or (11.120) with $\dot{\boldsymbol{\omega}} = 0$ and $\omega_3 = 0$, the torque needed is $\mathbf{N} = N_3\hat{\mathbf{e}}_3 = -(I_1 - I_2)\omega_1\omega_2\,\hat{\mathbf{e}}_3 = \frac{1}{12}mab\omega^2\left(\frac{b^2-a^2}{b^2+a^2}\right)\hat{\mathbf{e}}_3$.

6. [2 pt] In the class we discussed a special case of three-body problem called a *restricted* three-body problem.

(a) [1 pt] Describe the condition for a restricted three-body problem.

(b) [1 pt] We considered several examples which we can model as restricted three-body problems. Describe one or more such cases. 2-3 sentences per case are expected to clearly explain *(i)* which three bodies we are examining, and *(ii)* how the concepts of effective potential, equipotential surface (contour), and/or the Lagrange points are used. Circle these keywords clearly in your answer. Use diagrams if desired.

• (b) Man-made satellites at Sun-Earth L_1 and L_2, space colonies at Sun-Earth L_4 and L_5, Earth Trojan 2010 TK7 at Sun-Earth L_4, Jovian Trojans at Sun-Jupiter L_4 and L_5, gas flowing along the equipotential contour and/or filling the Roche lobe before the Type Ia supernova explosion. For more information about each of the items above, see the class slides, Lecture 8-1.