Classical Mechanics II (Fall 2020): Midterm Examination

Oct. 24, 2020

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from ① to ⑥ on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet ②). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

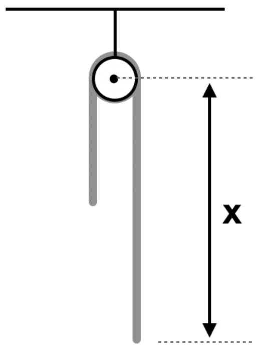
• Make your writing easy to read. Illegible answers will *not* be graded. When asked, circle the keywords in your answer clearly.

1. [3 pt] A very flexible rope of mass m, uniform density ρ and total length $2b$ hangs in equilibrium over a frictionless horizontal pin of negligible diameter. The rope is given a slight vertical displacement causing it to slowly roll off the pin. The rope is prevented from lifting off the pin, and only gravity acts on the rope. The length x of the longer portion of the rope will thus grow in time. The pin is positioned higher than $2b$ from the ground.

(a) [1 pt] Using energy conservation, show that the rope's velocity is written as $\dot{x} = \sqrt{\frac{g}{b}}(x - b)$ where g is the gravitational acceleration. Then find the rope's acceleration $\ddot{x}$ as a function of x.

(b) [1 pt] Find the rope's acceleration, this time using the Lagrangian method.

(c) [1 pt] Find the tension in the rope, F_T, at the top position right next to the pin.

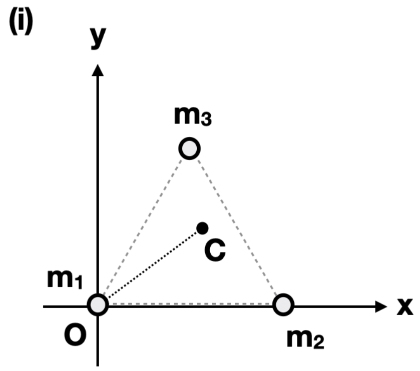


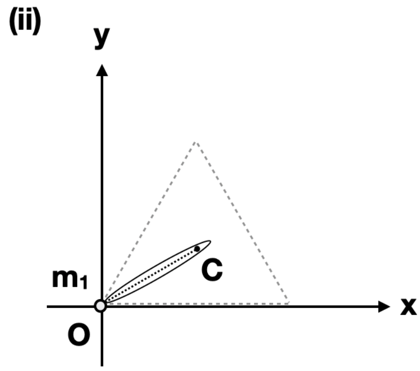
2. [4 pt] Consider a system of three stars with masses m_1, m_2 and m_3, placed at the corners of an equilateral triangle of side L (see figure (i) below). The total mass of the system is $M = m_1 + m_2 + m_3$. Only gravity acts between the bodies.

(a) [1 pt] First, assume that the stars are in a rotational motion under mutual gravitational attraction while keeping the relative separation of each pair of stars unchanged on the original triangle's plane. Using the coordinate seen below, find the location of the center of mass C.

(b) [1 pt] By demonstrating that the combined force acting on each star is directed toward C, show that each star is in a circular motion about C. Show that the period of the circular motions of all three stars is $T_{\text{circ}} = 2\pi\sqrt{\frac{L^3}{GM}}$.

(c) [2 pt] Now assume that the stars are of equal masses, and initially at rest. Show that the stars will collapse to the center of the triangle C with no tangential velocity, and collide with one another in time $T_{\text{col}} = \frac{1}{4\sqrt{2}} T_{\text{circ}} = \pi\sqrt{\frac{L^3}{8GM}}$. (Note: You may want to use the observation that a star collapsing from O to C traces a *severely* flattened elliptical orbit of a semi-major axis $\frac{\overline{OC}}{2}$. See figure (ii) below.)



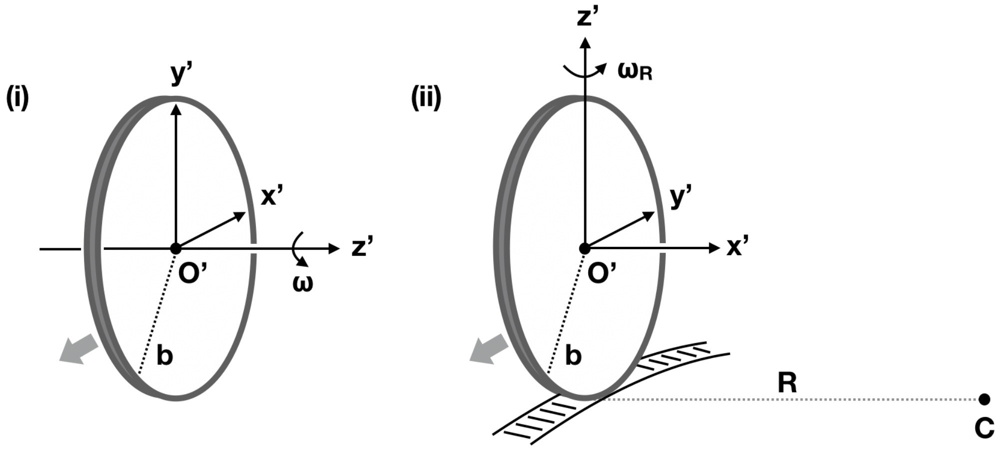


3. [4 pt] A car with wheels of radius b travels in the following ways. Find the acceleration, relative to the ground, of the highest point on one of its wheels. Depending on the problem setup, you may consider different types of rotating coordinate systems illustrated below (see next page).

(a) [1 pt] With constant forward velocity $\mathbf{v}_0$.

(b) [2 pt] With constant forward acceleration $\mathbf{a}_0$ and instantaneous velocity $\mathbf{v}$.

(c) [1 pt] With constant forward speed v_0 around a circular track of radius R.

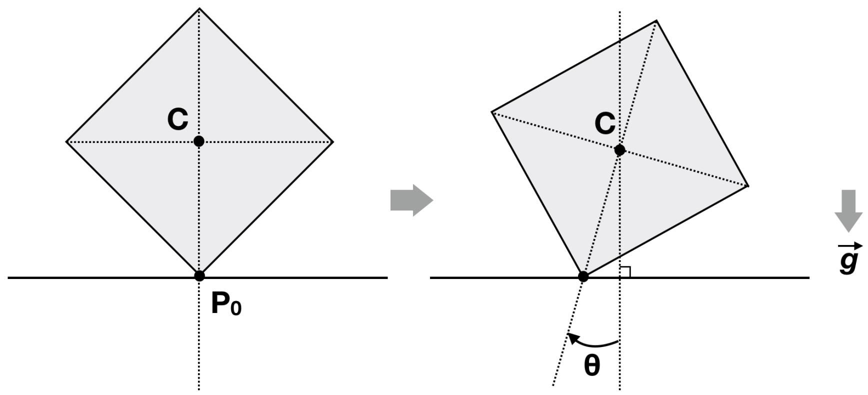


4. [4 pt] A homogenous cube, each edge of which has a length b, is initially in an unstable equilibrium with its one edge in contact with a horizontal plane. The cube is then given a slight nudge and allowed to fall in a uniform gravitational field g. P_0 is the point fixed on the ground that is right below the cube's center of mass, C, in the edge initially in contact with the plane.

(a) [1 pt] First, assume that the edge cannot slide on the plane. Find the angular acceleration of the cube's rotation as a function of θ (see the figure). You may first want to find the location of C with respect to P_0.

(b) [1 pt] Determine the angular velocity of the cube's rotation when one of its faces strikes the plane.

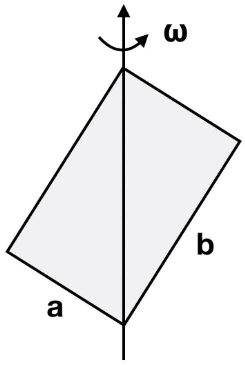
(c) [2 pt] Now assume that the plane is frictionless (i.e., sliding can occur without friction). Determine the angular velocity of the cube's rotation when one of its faces strikes the plane.



5. [3 pt] A thin rectangular plate with mass m and sides a and b is spinning around its diagonal with constant angular velocity ω.

(a) [1 pt] After defining your body coordinate system, find the angular momentum of the plate.

(b) [2 pt] Find the magnitude and direction of the torque **N** that needs to be applied to keep the plate spinning with constant ω. Describe what happens when $a = b$.



6. [2 pt] In the class we discussed a special case of three-body problem called a *restricted* three-body problem.

(a) [1 pt] Describe the condition for a restricted three-body problem.

(b) [1 pt] We considered several examples which we can model as restricted three-body problems. Describe one or more such cases. 2-3 sentences per case are expected to clearly explain *(i)* which three bodies we are examining, and *(ii)* how the concepts of effective potential, equipotential surface (contour), and/or the Lagrange points are used. Circle these keywords clearly in your answer. Use diagrams if desired.