

# Classical Mechanics I (Spring 2026): Homework #4

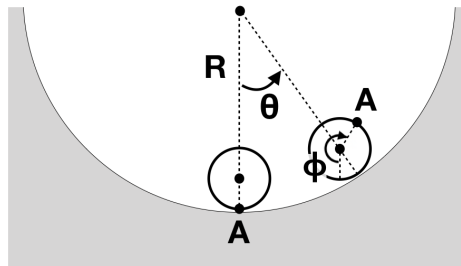
Due Jun. 1, 2026

[0.5 pt each, total 6 pts / turn in as a single pdf file to eTL]

- By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code, and the SNU AI Guideline!)
- Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.
- For some problems you may want to use formulae in Appendices D and E of Marion, and/or more extensive references such as Zwillinger.

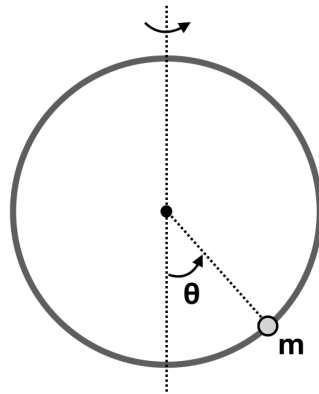
1.-8. Thornton & Marion, Problems 7-3, 7-18, 7-33, 7-34, 8-12, 8-22, 8-31, 8-33

(Note: For Problem 7-3, assume that the sphere oscillates in a plane perpendicular to the ground, and tackle the problem using three different approaches: *(i)* the Lagrangian method with undetermined multipliers, *(ii)* the Lagrangian method without undetermined multipliers, and *(iii)* the Hamiltonian method. Use the following variables:  $\theta$  is the angle between the vertical and the line joining the centers of the sphere and the cylinder, and  $\phi$  is the angle by which the sphere rotated from the bottom of the cylinder. For Problem 7-33, first explicitly derive Eqs.(7.56) and (7.57) using the Lagrangian method. Use  $m_1 = 5m$ ,  $m_2 = 3m$ , and  $m_3 = 2m$  for simplicity. For Problem 7-34(b), acquire the reaction using both Lagrangian and Newtonian methods. The “reaction” refers to the normal force of constraint exerted by the wedge on the particle, which of course depends on the initial position of the particle. For Problem 8-22, sketch the particle’s motion in the effective potential diagrams,  $V(r)$ , of various  $\ell$  and  $E$  values — i.e., equivalents of Figure 8-6. For Problem 8-33, you may want to use an approximation technique similar to what we did in Eq.(8.86) or (8.99).)



9. A bead of mass  $m$  is threaded on a frictionless, circular wire hoop of radius  $R$ . The hoop rotates about its vertical diameter with constant angular velocity  $\omega$ .

- (a) Determine Lagrange's equation(s) of motion of the bead.
- (b) Find Hamilton's equation(s) of motion of the bead. Is the Hamiltonian the total energy of the bead? (Note: For perspectives on this question, see Problem 7-17 of Thornton & Marion).
- (c) Determine the equilibrium position(s) of the bead, and find the critical angular speed  $\omega_c$  that delineates distinct types of behaviors of the system. For the *nonzero* equilibrium position(s) — i.e., away from the top or the bottom of the hoop — discuss the Newtonian force equilibrium.
- (d) Discuss the stability of each equilibrium position. If a stable equilibrium point exists, find the period of small oscillations about the point.
- (e) Draw the  $\omega - \theta_s$  diagram schematically, where  $\theta_s$  denotes the stable equilibrium position(s). Explain the diagram's bifurcation behavior as  $\omega$  increases.
- (f) Now drop the small oscillation assumption in (d), and numerically solve the nonlinear equation of motion for the nonzero stable equilibrium position(s). Plot your solution  $\theta(t)$  and compare it with the approximate solution from (d). For simplicity, one may plot the solution with  $g = 1$ ,  $R = 1$ ,  $\omega = \sqrt{2}$ ,  $\dot{\theta}(0) = 0$ , and a perturbation  $\theta(0) - \theta_s = 1^\circ$ . Experiment other perturbation values such as  $10^\circ$ . Does the small oscillation assumption work?



10. The Legendre transformation is a powerful mathematical tool used to change the functional dependence of a system from one set of variables to their conjugate variables, typically replacing an independent variable ( $x$ ) with the derivative of the function with respect to that variable ( $\frac{\partial f}{\partial x}$ ), while ensuring that no physical information is lost during the transformation.

- (a) This transformation is indispensable in classical mechanics for transitioning from the Lagrangian formalism  $L(q, \dot{q})$ , which depends on velocities, to the Hamiltonian formalism  $H(q, p)$ , which depends on momenta. We discussed this extensively in the class. Given  $L(q, \dot{q})$  with  $dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} = \frac{\partial L}{\partial q} dq + p d\dot{q}$  (note  $p \equiv \frac{\partial L}{\partial \dot{q}}$ ), one can find  $H(q, p)$  so that  $dH = \frac{\partial L}{\partial q} dq - \dot{q} dp$ . Find explicitly a Legendre transformation that gives  $H(q, p)$ .

Similarly, in thermodynamics, the Legendre transformation allows for the conversion between fundamental potentials, such as transforming the internal energy  $U(S, V)$  into the enthalpy  $H(S, P)$  or the Gibbs free energy  $G(T, P)$ .

(b) Given  $U(S, V)$  with  $dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV = TdS - PdV$  (note  $T \equiv \left(\frac{\partial U}{\partial S}\right)^{-1}$  and  $P \equiv -\frac{\partial U}{\partial V}$ ), one can find  $H(S, P)$  so that  $dH = TdS + VdP$ . Find a Legendre transformation that gives  $H(S, P)$ . Discuss the meaning of the two functions,  $U$  and  $H$ , by identifying them as (internal) energy and enthalpy, respectively.

(c) Given  $dU = TdS - PdV$ , perform a different Legendre transformation to find another useful function related to energy,  $F(T, V)$ , known as the Helmholtz free energy. Finally, by performing Legendre transformations on both terms in  $dU$ , find yet another function related to energy,  $G(T, P)$ , known as the Gibbs free energy.

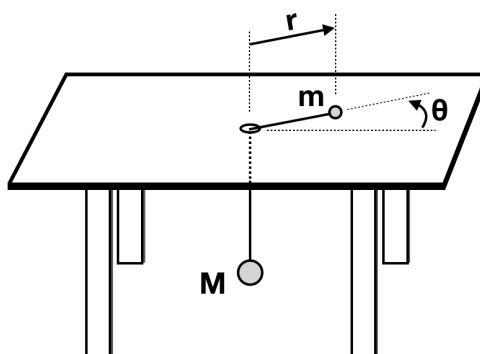
(Note: You may want to briefly review the relevant part in mathematical physics textbook such as Boas Chapter 4, Section 11. For (b) and (c), you may also want to review the textbooks in statistical mechanics such as Schroeder or Reif. In (b), you are simply asked to come up with 2-3 sentences about what the quantities  $U$  and  $H$  mean in statistical mechanics, without going into all the thermodynamical details. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see <http://library.snu.ac.kr/using/proxy>.)

11. A particle of mass  $m$  is connected to another particle of mass  $M$  via a massless string of fixed length  $b$  that passes through a small hole in a table. The particle  $m$  is free to slide on the frictionless table, while the other particle  $M$  hangs below the table and moves only vertically. The particles are subject to the gravitational field  $g$ . The string always remains tight.

(a) Determine the Lagrangian and Lagrange's equations of motion for the generalized coordinates  $r$  and  $\theta$  shown in the figure (note:  $\theta$  is measured on the plane of the table).

(b) Now, by writing Lagrange's equation(s) of motion with undetermined multipliers and the equation(s) of constraints, express the tension of the string in terms of the variables given and their time derivatives.

(c) Find the condition under which  $m$  makes circular motion at  $r = r_0 = \text{constant}$ . Is this motion stable? What is the period of small oscillations (in the variable  $r$ ) about this circular motion?



[Problem 12 continues in the next page.]

12. A particle of mass  $\mu$  moves with angular momentum  $\ell$  in a central-force field described by  $F(r) = -\frac{k}{r^{5/2}}$ . This problem needs to be solved numerically. For simplicity, one may choose units for which  $\mu$ ,  $\ell$  and  $k$  all become 1.

(a) Draw the effective potential  $V(r)$  and find the radius of a stable circular orbit.

(b) Assuming now that the particle has energy  $E = -0.1$ , find the particle's distance of closest approach to the force center,  $r_{\min}$ .

(c) Solve the *radial* equation of motion for  $E = -0.1$ , and plot your solution  $r(\theta)$ . Assume that the particle is at  $r = r_{\min}$  when  $\theta = 0$ . Can you determine if the orbit is closed?

13. [For additional +0.5pt] Here we review the sections of Thornton & Marion we briefly discussed in the class and left for your exercise.

(a) In the class we discussed the connection between invariance or symmetry properties and conserved quantities (Noether's theorem). Starting from three characteristics of an inertial reference frame, follow step by step the logical procedures that eventually lead to Eqs.(7.130), (7.137) and (7.149). During the process, you will need to carefully derive Eq.(7.122), and also need to show that an expression like Eq.(1.106) is possible for infinitesimal rotation.

(b) Study Liouville's theorem, Eq.(7.198), and the virial theorem, Eq.(7.204). For example, you may follow step by step the procedures in your textbook that eventually proves each theorem. These two theorems have important applications in statistical mechanics and in astrophysics. Explain briefly.