Classical Mechanics I (Spring 2026): Midterm Examination

Apr. 18, 2026

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from (1) to (6) on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet (2)). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

• Make your writing easy to read. Illegible answers will *not* be graded.

• Do *not* use the Lagrangian method.

1. [3 pt] Consider a damped oscillator of mass m subject to a restoring force $-kx$. Its motion is described by the equation $\ddot{x}+2\beta\dot{x}+\omega_0^2 x = 0$, where $x(t)$ is the displacement at time t, $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency, and $\beta = \frac{b}{2m}$ is the damping constant.

(a) [2 pt] By explicitly solving this 2nd-order homogeneous ordinary differential equation, find the general solution for the case $\beta < \omega_0$. Express your answer in a sinusoidal form, while introducing new constants if needed.

(b) [1 pt] The amplitude of a damped oscillator decreases to $1/e$ of its initial value after n complete oscillation cycles. Show that the ratio of the oscillation period to that of the corresponding undamped oscillator is approximately $\left[1 + (8\pi^2 n^2)^{-1}\right]$. (Note: You may assume that the damping is weak, so that n is significantly larger than 1.)

2. [3 pt] Consider the same damped oscillator as in Problem 1, but now driven by an external sinusoidal force. Its motion is then described by the equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$.

(a) [2 pt] By explicitly solving this 2nd-order inhomogeneous ordinary differential equation, find the steady-state solution. Express your answer in a sinusoidal form, $D(\omega)\cos(\omega t - \phi(\omega))$, where the amplitude $D(\omega)$ and the phase difference $\phi(\omega)$ are functions of the driving frequency ω.

(b) [1 pt] Construct the electrical analog of this system and calculate the magnitude of its impedance.

3. [4 pt] A particle of mass m moves in one-dimension in the region $x > 0$ under the force $F(x) = -k\left(x - \frac{x_0^4}{x^3}\right)$, where k and x_0 are positive constants .

(a) [1 pt] Sketch the potential energy $U(x)$.

(b) [2 pt] Identify all equilibrium point(s) on the $U(x)$ plot and discuss their stabilities. Determine the period of small oscillations about the stable equilibrium point(s).

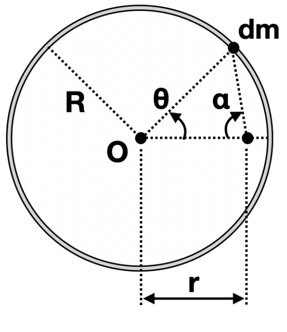
(c) [1 pt] Now abandon the assumption of small oscillations. Show that, even for arbitrarily large energies, the period of oscillation about the stable equilibrium point found in (b) is independent of the energy. (Note: If you encounter an integral of the form $\int \frac{dx}{\alpha x^2+\beta+\gamma x^{-2}}$, consider the substitution $u = x^2$. You may then utilize one of the tabulated integrals, Eq.(E.8c) in Appendix E of Thornton & Marion,

$$\int \frac{du}{\sqrt{A^2 - (u-u_0)^2}} = \sin^{-1}\left(\frac{u-u_0}{A}\right),$$

valid for $|u - u_0| < A$. This problem can be algebraically demanding; manage your time wisely.)

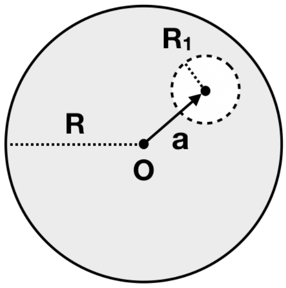
4. [4 pt] Consider a thin, uniform spherical shell of mass M and radius R.

(a) [2 pt] Show that the gravitational force on a test particle inside the shell is zero, by directly computing the gravitational field vector $\mathbf{g}(r)$ as a function of distance r from the center of the shell, O. Consider only gravitational force, and assume no other gravitational source.



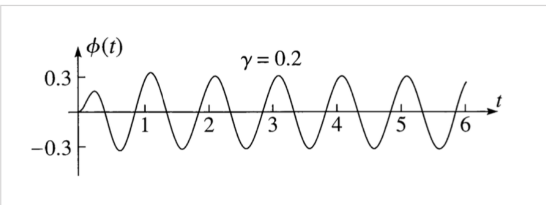
(b) [1 pt] Repeat (a), now by computing the gravitational potential $\Phi(r)$.

(c) [1 pt] A sphere of uniform density ρ and radius R has within it a spherical cavity of radius R_1 whose center is at $\mathbf{a}$ as seen in the figure below ($\mathbf{a}$ originates from the center of the sphere O and points to the center of the cavity O_1). After first establishing the gravitational field inside a filled sphere of uniform density, show that the gravitational field within the cavity is uniform. Determine both its magnitude and its direction.

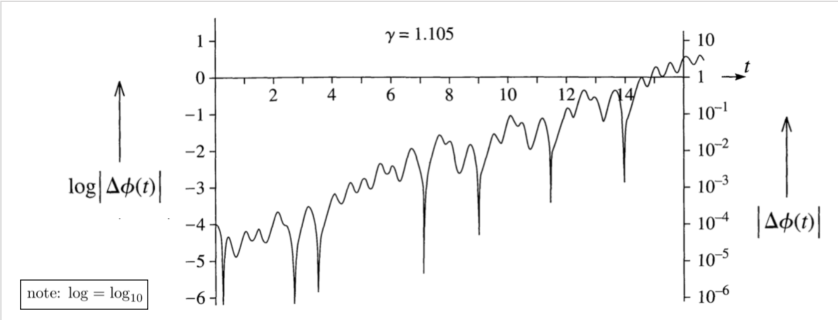


5. [3 pt] Consider a damped driven plane pendulum described by $\ddot{\phi}+2\beta\dot{\phi}+\omega_0^2\sin\phi = \gamma\omega_0^2\cos\omega t$, where $\phi(t)$ is the angular displacement in radian at time t, ω is the driving frequency, $\omega_0 = 1.5\,\omega$ is the natural angular frequency of the pendulum, and $\beta = 3\,\omega/8 = 0.25\,\omega_0$ is the damping constant. For simplicity, we choose the driving frequency to be $\omega = 2\pi$ so that the drive cycle τ becomes 1. Now let us consider two identical pendulums that satisfy the exact same equation of motion, but have slightly different initial conditions. We denote the separation of these two pendulums as $\Delta\phi(t) = \phi_2(t) - \phi_1(t)$.

(a) [0.5 pt] First, consider the linear regime in which $\sin\phi \approx \phi$ (e.g., when the initial velocity $\dot{\phi}(0)$ and the drive strength γ are small). Using the figure below, describe the behavior of $|\Delta\phi(t)|$ as $t \to \infty$. In your answer, include and *circle* the following keywords: complementary solution, particular solution, transient effect, steady state, initial conditions. (Note: You don't need to solve differential equations here; qualitative justifications are more than enough.)



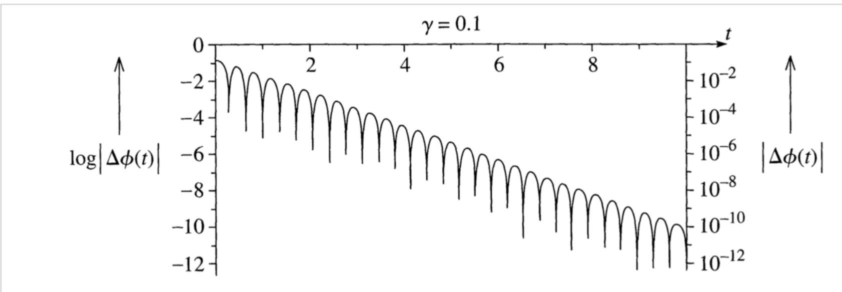
(b) [0.5 pt] From now on, we consider the nonlinear regime in which the approximation $\sin\phi \approx \phi$ no longer holds. Different behavior may emerge for different γ. For example, for $\gamma = 1.105$, $|\Delta\phi(t)|$ starts out at 10^{-4} but reaches π by $t = 16$, as seen in the figure below. Why can $t = 16$ be considered an important milestone?



(c) [0.5 pt] Use the observation in (b) to estimate the Lyapunov exponent λ, as defined in $\Delta\phi(t) \sim |\Delta\phi(0)|e^{\lambda t}$. Then, suppose that you want to predict the pendulum's $\phi(t)$ with an accuracy of 10^{-1} and that you know the initial value $\phi(0)$ within 10^{-6}. Estimate the maximum time T_{max} for which you can predict $\phi(t)$ within the required accuracy. (Note: Here "$\sim$" signifies that $\Delta\phi(t)$ on average oscillates roughly underneath the envelope $|\Delta\phi(0)|e^{\lambda t}$. Obtain your answers accurate to only two significant figures. Use the following if necessary: $\log(\pi) = \log_{10}(\pi) \approx 0.50$, $\log(e) = \log_{10}(e) \approx 0.43$.)

(d) [1 pt] If you wish to double T_{max} found in (c), by how much must the accuracy of your measurement of $\phi(0)$ be improved (e.g., to within $10^{-7}, 10^{-8}, \ldots$)? Use this example to explain why making accurate long-term predictions is inherently difficult for chaotic systems.

(e) [0.5 pt] Compare qualitatively the two cases, $\gamma = 1.105$ (figure above) and $\gamma = 0.1$ (figure below), focusing on the evolution of $|\Delta\phi(t)|$. In your answer, include and *circle* the following keywords: exponential decay/growth, sensitivity to initial conditions, linear, chaotic. (Note: You don't need to solve differential equations here; qualitative justifications are more than enough.)



6. [3 pt] In the class we covered several special topics including dark matter and tidal forces.

(a) [2 pt] We discussed how the tidal forces manifest in various physical settings. Describe how the tidal interaction between the Earth and the Moon leads to each of the phenomena listed below. For each phenomenon, 3-4 sentences are expected that clearly explain, step by step, how the concept of tidal force accounts for the observed behavior. Use diagrams if desired.

- The Moon's rotation period is equal to its orbital period (i.e., tidal locking).
- The Moon has been moving away from the Earth at a rate of approximately $3.8\ \text{cm}\,\text{yr}^{-1}$.

(b) [1 pt] We also discussed several evidences that suggest the existence of dark matter. Describe two or more of them. 3-4 sentences per evidence are expected to clearly explain how each evidence points to the existence of dark matter. Use diagrams if desired.