Classical Mechanics I (Spring 2020): Homework #2 Solution

Due Apr. 28, 2020

[0.5 pt each, total 7 pts]

1. Thornton & Marion, Problem 4-4

• Plugging $\dot{x} = \sqrt{\frac{a}{3b}}\frac{y}{y_0}$, $\ddot{x} = \sqrt{\frac{a}{3b}}\frac{\dot{y}}{y_0}$, and $\dddot{x} = \sqrt{\frac{a}{3b}}\frac{\ddot{y}}{y_0}$ into the differentiated Rayleigh's equation gives van der Pol's equation.

2. Thornton & Marion, Problem 4-10

(Note: For Problem 4-10, drawing a phase diagram and a Poincaré section for each ω will help you appreciate if the system exhibits chaotic behavior or not. You do not have to include all the plots from your investigations in your answer, but include only a few representative one.)

• Among the ω values suggested in the problem, only $\omega = 0.6$ and 0.7 shows chaotic behaviors in a foreseeable time period for which your computer can handle the calculation.

3. Thornton & Marion, Problem 4-14

• At $n = 30$, $|\Delta x_n| = |x_{n,\, x_1=0.9} - x_{n,\, x_1=0.9000001}|$ starts to become larger than 30% of $|x_{n,\, x_1=0.9}|$.

4. Thornton & Marion, Problem 4-17

• Only $\alpha = 0.7$ makes the tent map show chaotic behavior.

5. Thornton & Marion, Problem 4-19

• The Lyapunov exponent λ becomes $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln \left| \left[\frac{df(x)}{dx}\right]_{x_i} \right| = \lim_{n\to\infty} \frac{n-1}{n} \ln 2\alpha = \ln 2\alpha$ from Eqs.(4.51) and (4.52). As seen in Problem 4-17, only $\alpha = 0.7$ gives $\lambda > 0$.

6. Thornton & Marion, Problem 5-4

• From energy conservation, $\dfrac{1}{2}m\dot{x}^2 = \dfrac{mk^2}{2x^2} - \dfrac{mk^2}{2d^2} \quad \rightarrow \quad \displaystyle\int_0^T dt = -\frac{d}{k}\int_d^0 \frac{x\,dx}{\sqrt{d^2-x^2}} = \frac{d}{k}\left[\sqrt{d^2-x^2}\right]_d^0$

7. Thornton & Marion, Problem 5-10

• Slightly modified from Problem 5-9 discussed in the class,

$$\Phi(\mathbf{R}) = -G\int_V \frac{\rho(\mathbf{r}')dv'}{|\mathbf{R}-\mathbf{r}'|} = -\rho_l G\int_0^{2\pi}\frac{a\,d\phi}{x} = -\frac{GM}{2\pi R}\int_0^{2\pi}\left[1+\left(\frac{a}{R}\right)^2 - 2\left(\frac{a}{R}\right)\sin\theta\cos\phi\right]^{-\frac{1}{2}} d\phi$$

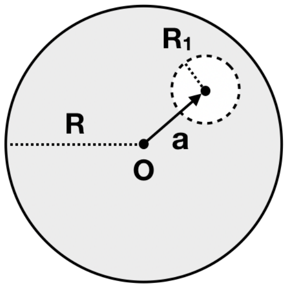
$$\approx -\frac{GM}{R}\left[1-\frac{1}{2}\left(\frac{a}{R}\right)^2\left(1-\frac{3}{2}\sin^2\theta\right)\right] \quad \text{if } R \gg a$$

• As expected, the answer here reduces to the solution of Problem 5-9 when $\theta = 0$.

8. Thornton & Marion, Problem 5-16

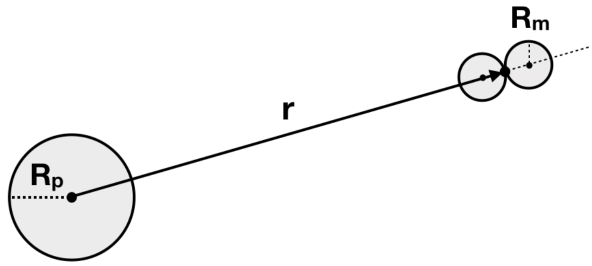
• Using the result of Example 5.4, the force on the sheet by the sphere M can be acquired by setting $a \to \infty$ and changing the sign of Eq.(5.47). Thus, $F_z = +2\pi M\rho G$.

9. A sphere of uniform density ρ and radius R has within it a spherical cavity of radius R_1 whose center is at $\mathbf{a}$ as seen in the figure below ($\mathbf{a}$ originates from the center of the sphere O and points to the center of the cavity O_1). Show that the gravitational field within the cavity is uniform. Determine its magnitude and direction.



• Using the principle of superposition and Eq.(5.22), the gravitational field vector $\mathbf{g}$ at an arbitrary location inside the cavity (the location denoted as $\mathbf{r}$ originating from O, or as $\mathbf{r}_1$ originating from O_1) is $\mathbf{g}(\mathbf{r}) = -\frac{4}{3}\pi G\rho\,\mathbf{r} - \frac{4}{3}\pi G(-\rho)\,\mathbf{r}_1 = -\frac{4}{3}\pi G\rho\,(\mathbf{a}+\mathbf{r}_1) + \frac{4}{3}\pi G\rho\,\mathbf{r}_1 = -\frac{4}{3}\pi G\rho\,\mathbf{a}$.

10. An oddly-shaped moon orbiting a massive exoplanet at radius r can be modeled as two identical spheres of uniform density $\rho_{\rm m}$ and radius $R_{\rm m}$ ($\ll r$) just touching each other. Their centers are always in line with the center of the planet whose mean density is $\rho_{\rm p}$ and radius $R_{\rm p}$. The only force between the three objects is gravitational. Show that the two spheres will be pulled apart by the planet's tidal force if the moon is less than $r_{\rm lim} = 2\left(\frac{\rho_{\rm p}}{\rho_{\rm m}}\right)^{\frac{1}{3}} R_{\rm p}$ away from the planet's center. (Note: Assume that the two spheres themselves cannot be torn apart.)



- From Eq.(5.52) at the critical radius $r_{\rm lim}$, $\dfrac{2G\left(\frac{4}{3}\pi R_{\rm p}^3\rho_{\rm p}\right)\left(\frac{4}{3}\pi R_{\rm m}^3\rho_{\rm m}\right)R_{\rm m}}{r_{\rm lim}^3} = \dfrac{G\left(\frac{4}{3}\pi R_{\rm m}^3\rho_{\rm m}\right)^2}{(2R_{\rm m})^2}$

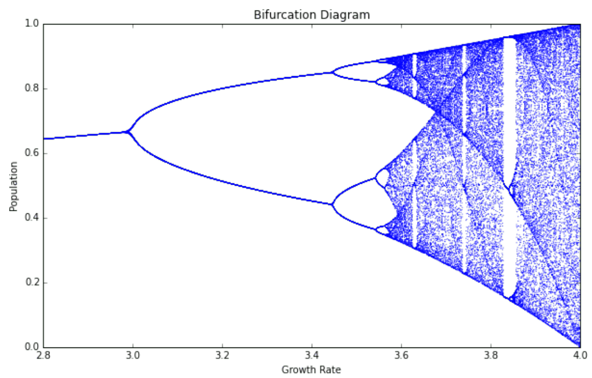
11. [1 pt] In this problem, let us make the bifurcation diagram of the logistic map, Eq.(4.46).

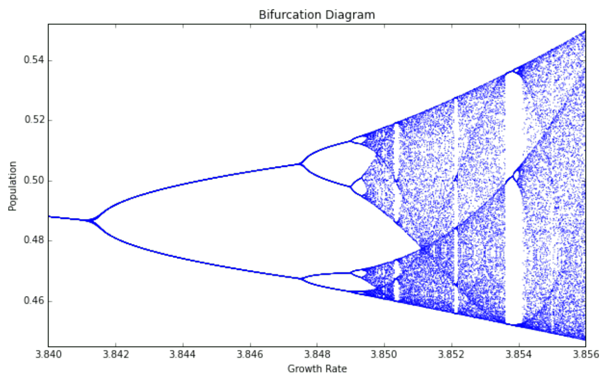
(a) First, reproduce Figure 4-23. Start testing your script with a small number of points, e.g., α going from 2.8 to 4.0 in steps of 0.2. Once you are ready to make a final diagram, you may increase the number of points, e.g., α going from 2.8 to 4.0 in steps of 0.01 or less.

(b) Make a zoomed-in version of the bifurcation diagram from (a) in a small rectangular region of $\alpha \in [3.840, 3.857]$ and $x_n \in [0.44, 0.55]$. You may want to use much finer steps in α, of course. Notice the emerging self-similarity, though the new diagram is upside down in a smaller scale.

(c) Thornton & Marion, Problem 4-11

- See e.g., geoffboeing.com/2015/03/chaos-theory-logistic-map.





12. Choose a topic in any scientific research — physics, biology, epidemiology, economics, human behavioral science, etc. — in which the concepts we newly learned in Chapter 4 of Thornton & Marion are used. Describe how a complicated and seemingly random phenomenon can be modeled with the concepts in nonlinear dynamics or in chaos theory. Use diagrams if desired.

(Note: 2-3 paragraphs or more per phenomenon are expected to clearly unfold your story. You must reference your sources appropriately with a proper citation convention, and your answer must be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

13. In the class we discussed the method of Green's function. Starting from a response to a Heaviside step function, Eq.(3.100), follow step by step the logical procedure that eventually leads to Eq.(3.118). In particular, carefully extract Eq.(3.110) by combining Eq.(3.108) and the limiting case assumptions – which was briefly discussed in the class and left for your exercise.