Classical Mechanics I (Spring 2020): Homework #1 Solution

Due Apr. 7, 2020

[0.5 pt each, total 6 pts]

1. Thornton & Marion, Problem 2-7

(Note: For Problems 2-7, you will need to use Eq.(2.21) with $\rho_{\text{air}} = 1.23\ \text{kg}\,\text{m}^{-3}$.)

• $m\ddot{x} = -\frac{1}{2}c_W \rho A \dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad m\ddot{y} = -mg - \frac{1}{2}c_W \rho A \dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$

• The bales should be released ∼180 m behind the cattle as opposed to ∼210 m you saw in the in-class quiz, Problem 2-6(a).

2. Thornton & Marion, Problem 2-15

(Note: For Problem 2-15, you may need to utilize an integral table; see the third bullet point in the problem set.)

• $m\dot{v} = mg\sin\theta - kmv^2 \quad \rightarrow \quad \frac{dv}{v^2-(g/k)\sin\theta} = -kdt \quad \rightarrow \quad \tanh^{-1}\left(\frac{v}{\sqrt{(g/k)\sin\theta}}\right) = \sqrt{gk\sin\theta}\cdot t$
where we utilized Eq.(E.4c) in Appendix E.

3. Thornton & Marion, Problem 2-22

• (b) $\ddot{x} = \omega_c \dot{y}, \quad \ddot{y} = -\omega_c(\dot{x} - E_y/B), \quad \ddot{z} = qE_z/m$

• (c) $\dddot{x} = -\omega_c^2(\dot{x} - E_y/B) \quad \rightarrow \quad \dot{x} = C_1\cos\omega_c t + C_2 \sin\omega_c t + E_y/B$

4. Thornton & Marion, Problem 2-40

• (a) $a_t = \dot{v} = \frac{d(\sqrt{\dot{x}^2+\dot{y}^2})}{dt} = \frac{2A\alpha^2\sin\alpha t}{\sqrt{5-4\cos\alpha t}}$

• (b) $a_n = \sqrt{a^2 - a_t^2}$ has the maximum value when $t = \frac{N\pi}{\alpha}$ where N is an integer.

5. Thornton & Marion, Problem 2-47

• At $x = x_0 = a$, $[d^2U/dx^2]_{x=x_0} = 2U_0/a^3 > 0$

6. Thornton & Marion, Problem 3-10

• From Eq.(3.38), $\omega_1^2 = \omega_0^2 - \beta^2 = \omega_0^2 - \left(\frac{\omega_1}{2\pi n}\right)^2$

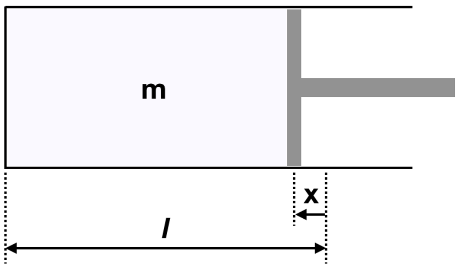
7. Thornton & Marion, Problem 3-22

• (a) Inserting the initial conditions to Eq.(3.44), one gets $x_0 = A_1 + A_2$ and $v_0 = -A_1\beta_1 - A_2\beta_2$.

8. Thornton & Marion, Problem 3-24

• The amplitude of oscillation, Eq.(3.59), of the particular solution $x_p(t)$ peaks at ω_R, Eq.(3.63).

9. Gas of mass m is confined by a frictionless, massless piston in a cylinder of cross-section A. The gas in its equilibrium position occupies a volume $V = Al$ at pressure P. If the piston is slightly compressed inwards (i.e., $l \to l - x$) under isothermal conditions, find the period of simple harmonic oscillations.



• $P'A(l-x) = PAl$ if isothermal $\to$ $P' \simeq P\left(1 + \frac{x}{l}\right)$ if $\frac{x}{l} \ll 1$

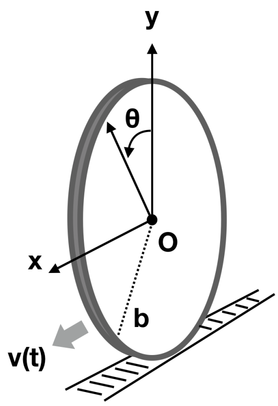
• The restoring force is then $F(x) = -(P' - P)A = -\frac{PA}{l}x = m\left(\frac{\ddot{x}}{2}\right)$ $\to$ $T = 2\pi\sqrt{\frac{ml}{2PA}}$ (An order of magnitude estimation would be sufficient; difference by a factor of a few would be okay.)

10. A wheel of radius b travels with constant forward acceleration a_0 and instantaneous forward velocity $v(t)$ relative to the ground at time t.

(a) Show that the magnitude of the acceleration of any point on the rim relative to the center of the wheel O is $\sqrt{a_0^2 + \frac{v^4(t)}{b^2}}$.

(b) Find the magnitude of the acceleration of point P on the rim (defined by an angle θ measured forward from the highest point of the wheel) relative to the ground.

(c) Which point has the maximum acceleration relative to the ground at given t?

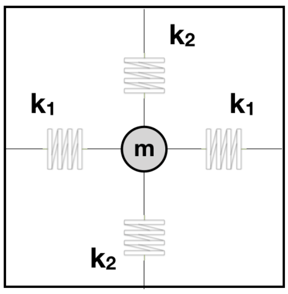


• (a) Let us define $\mathbf{v}_X(t)$ as the velocity of point X relative to the ground. Then, $\mathbf{v}_O(t) = v(t)\hat{\mathbf{e}}_x$ and $\mathbf{v}_P(t) = (v(t) + b\dot{\theta}\cos\theta)\,\hat{\mathbf{e}}_x - b\dot{\theta}\sin\theta\,\hat{\mathbf{e}}_y$, where $b\dot{\theta} = v(t)$. From this, one can show that the derivative of the relative velocity between the two points O and P has the magnitude of $\left|\frac{d}{dt}(\mathbf{v}_P(t) - \mathbf{v}_O(t))\right| = \sqrt{a_0^2 + \frac{v^4(t)}{b^2}}$ independent of θ.

• (b) $\left|\frac{d}{dt}(\mathbf{v}_P(t))\right| = a_0\sqrt{2 + 2\cos\theta + \frac{v^4(t)}{a_0^2 b^2} - \frac{2v^2(t)\sin\theta}{a_0 b}}$

• (c) By plotting the above function and/or by finding its first and second derivatives, one can find that the acceleration peaks at $\theta = 2\pi - \tan^{-1}\left(\frac{v^2(t)}{a_0 b}\right)$.

11. In the class we discussed two-dimensional simple harmonic oscillators. (a) Describe an example of the physical realization of Eqs. (3.19) and (3.27) using the figure shown below. (b) Reproduce Figures 3-2 to 3-4 in your textbook with your favorite numerical tool. For Figure 3-2, you don't have to remake all 10 panels; 2-3 panels should be enough.



• A mass m is tied to four springs, two of constant k_1 in the x-direction, and two others of constant k_2 in the y-direction. One end of each of the springs is tied to m (in a + shape), and the other end is tied to walls. At equilibrium none of the springs is stretched. We consider only small oscillations from the equilibrium point.

12. Consider a pendulum of length ℓ and a bob of mass m at its end. Assuming a small angular amplitude A, calculate the "average" tension (averaged over time) in the pendulum's string. Discuss the meaning of your answer by comparing it with mg.

• Equations of motion: $m\ell\ddot{\theta} = -mg\sin\theta$ in $\hat{\mathbf{e}}_\theta$, and $0 = T - mg\cos\theta - m\ell\dot{\theta}^2$ in $\hat{\mathbf{e}}_r$. Inserting $\theta = A\cos\omega t$ and $\cos\theta \approx 1 - \frac{A^2}{2}\cos^2\omega t$, one gets $\langle T\rangle \approx mg\left(1 + \frac{A^2}{4}\right)$ which is bigger than mg.