Classical Mechanics I (Spring 2020): Final Exam Solution

Jun. 20, 2020

[total 25 pts, closed book/cellphone, no calculator, 90 minutes]

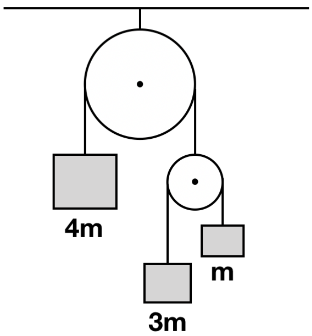
1. [4 pt] Consider a double pulley system shown below with massless inextensible strings and two massless smooth pulleys. The masses of the three weights are indicated in the figure (m, $3m$, $4m$).

(a) [1 pt] Determine the Lagrangian and Lagrange's equations of motion (without using undetermined multipliers). You will first need to define your choice of the generalized coordinates.

(b) [1 pt] Determine the downward acceleration of each of the three masses.

(c) [1 pt] Repeat (a) using the Lagrangian method with undetermined multipliers. Identify the forces of constraint and explain their physical meanings.

(d) [1 pt] Repeat (a) using the Newtonian method.



• (a) x_1: distance between the center of mass of the larger pulley (pulley 1) and $4m$, x_2: distance between the center of mass of the smaller pulley (pulley 2) and $3m$, then, $4m\ddot{x}_1 + 3m(\ddot{x}_1 - \ddot{x}_2) + m(\ddot{x}_1 + \ddot{x}_2) = 0$ and $-3m(\ddot{x}_1 - \ddot{x}_2) + m(\ddot{x}_1 + \ddot{x}_2) = 2mg$ (discussed in the class or in Example 7.8)

• (b) The downward accelerations of the masses $4m$, $3m$, m are $\frac{1}{7}g$, $\frac{3}{7}g$, $-\frac{5}{7}g$, respectively.

• (c) The Lagrange multipliers are related to the strings' tensions, $T_1 = \frac{24}{7}mg$, $T_2 = \frac{12}{7}mg$.

2. [4 pt] Show that the shortest path between two given points on the surface of a sphere is a *great circle* – in three different ways below.

(a) [2 pt] By introducing a functional in the form of $\int f_1\{\theta, \theta'\}\, d\phi$ in spherical coordinates $[r, \theta, \phi]$. One may opt to use the formula $\int \frac{a\csc^2\theta d\theta}{\sqrt{1-a^2\csc^2\theta}} = \sin^{-1}\left(\frac{\cot\theta}{\sqrt{(1/a^2)-1}}\right) + \text{constant}$ (when $a < 1$) to expedite the derivation.

(b) [1 pt] By using an alternative functional form of $\int f_2\{\theta, \phi'\}\, d\theta$.

(c) [1 pt] By rewriting the functional in the form of $\int f_3\{y', z'\}\, dx$ in Cartesian coordinates $[x, y, z]$, and employing the method of Euler equations with an auxiliary condition imposed.

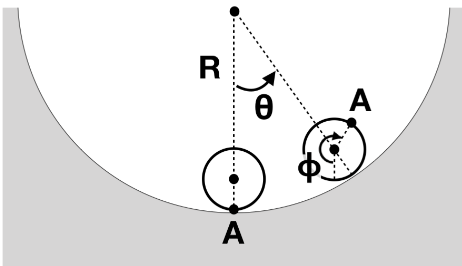
- (a) $s = \rho \int_1^2 \sqrt{\theta'^2 + \sin^2\theta}\, d\phi$ (discussed in the class or in Example 6.4)
- (b) $s = \rho \int_1^2 \sqrt{1 + \sin^2\theta\phi'^2}\, d\theta \;\rightarrow\; \frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = b \;\rightarrow\; \frac{d\phi}{d\theta} = \frac{b\csc^2\theta}{(1-b^2\csc^2\theta)^{1/2}}$: same as Eq.(6.46)
- (c) $s = \int_1^2 \sqrt{1 + y'^2 + z'^2}\, dx$ and $g = x^2 + y^2 + z^2 - \rho^2 = 0 \;\rightarrow\; \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}}\right) - 2\lambda(x)y = 0$ and $\frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}}\right) - 2\lambda(x)z = 0 \;\rightarrow\;$ Eliminating λ, $(z - xz')y'' - (y - xy')z'' = 0$.

3. [5 pt] Consider a homogenous disk of radius ρ and mass m, which rolls without slipping inside the lower half of a fixed, hollow cylinder of inner radius R. As indicated in the figure below, θ is the angle between the vertical and the line joining the centers of the disk and the cylinder, and ϕ is the angle by which the disk rotated from the bottom of the cylinder.

(a) [2 pt] Find the period of small oscillations in the variable θ using the Lagrangian method (without undetermined multipliers).

(b) [1 pt] Repeat (a) using the Lagrangian method with undetermined multipliers.

(c) [2 pt] Repeat (a) using the Hamiltonian method.



- (a) $T = \frac{1}{2}m(R-\rho)^2\dot{\theta}^2 + \frac{1}{4}m\rho^2\left(\frac{(R-\rho)\dot{\theta}}{\rho}\right)^2$ because $R\theta = \rho(\theta + \phi)$, and $U = -mg(R-\rho)\cos\theta \;\rightarrow\; \ddot{\theta} + \frac{2g}{3(R-\rho)}\sin\theta = 0$
- (b) $T = \frac{1}{2}m(R-\rho)^2\dot{\theta}^2 + \frac{1}{4}m\rho^2\dot{\phi}^2$, $U = -mg(R-\rho)\cos\theta$, and $g(\theta, \phi) = (R-\rho)\theta - \rho\phi = 0$

• (c) From L in (a), $p_\theta = \frac{\partial L}{\partial \dot\theta} = \frac{3}{2}m(R-\rho)^2\dot\theta \rightarrow H = p_\theta\dot\theta - L = p_\theta^2\left[3m(R-\rho)^2\right]^{-1} - mg(R-\rho)\cos\theta \rightarrow \dot\theta = \frac{\partial H}{\partial p_\theta} = 2p_\theta\left[3m(R-\rho)^2\right]^{-1}$ and $\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -mg(R-\rho)\sin\theta \rightarrow$ combined, $\dot{p_\theta} = \left[\frac{3}{2}m(R-\rho)^2\right]\ddot\theta = -mg(R-\rho)\sin\theta \rightarrow \ddot\theta + \frac{2g}{3(R-\rho)}\sin\theta = 0$

4. [3 pt] Throughout the semester we discussed many examples in which concepts in classical mechanics are utilized in contemporary research and in explaining daily phenomena.

(a) [2 pt] We discussed how invisible objects in the Universe were discovered by modeling a two-body system interacting via the central force. Describe three or more such cases. 2-3 sentences per case are expected to clearly explain how people came to notice the existence (and mass) of the invisible object using the concept of two-body motion, central force, and/or reduced mass. Use diagrams if desired.

(b) [1 pt] In the last class of the semester, five of your peers presented their term projects. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 2-3 sentences is expected to clearly convey the core physics idea of his/her term project. If you were one of the presenters, please choose someone else's.

• (a) Exoplanet systems, Pluto+Charon system, Sirius A+B system, Cygnus X-1 system, supermassive black hole and nearby stars. For more information about each of the items above, see the class slides, Lectures 3-1 and 15-1.

• (b) See the student presentation slides for Lecture 16-1 that include the collection of term project presentations by five students on June 16.

5. [4 pt] Consider a particle of mass μ moving under the influence of an attractive central force of the form $F(r) = -\frac{k}{r^2}\exp\left(-\frac{r}{a}\right)$ where r is the distance from the origin of the central force, and k and a are positive constants.

(a) [2 pt] By using the concept of an effective potential, find the condition for radius r_0 at which the particle with angular momentum ℓ moves in a circular orbit.

(b) [2 pt] Then, again using the concept of the effective potential, show that, if $r_0 < a$, a slight radial nudge to this circular orbit causes only small radial oscillations. Find the period for these oscillations.

• (a) $\left.\frac{\partial V(r)}{\partial r}\right|_{r_0} = 0$ with $V(r) = -\frac{ka}{r^2}\exp\left(-\frac{r}{a}\right) + \frac{\ell^2}{2\mu r^2} \rightarrow \frac{k}{r_0^2}\exp\left(-\frac{r_0}{a}\right) = \frac{\ell^2}{\mu r_0^3}$.

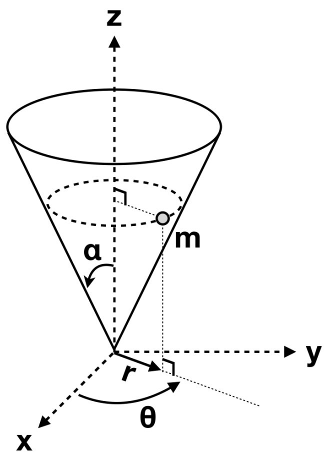
• (b) For the stability analysis, we use Eq.(8.76) or (8.93): $\left.\frac{\partial^2 V(r)}{\partial r^2}\right|_{r_0} = \frac{\ell^2}{\mu r_0^4}\left(1 - \frac{r_0}{a}\right) > 0$ for the orbit to be stable. $\omega_{\rm osc} = \sqrt{\frac{[\partial^2 V(r)/\partial r^2]_{r_0}}{\mu}} = \frac{\ell}{\mu r_0^2}\sqrt{1 - \frac{r_0}{a}}$ which is valid if $r_0 < a$.

6. [5 pt] A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α (see the figure below). The particle is subject to the gravitational field g.

(a) [2 pt] Determine the Lagrangian and Lagrange's equations of motion for the coordinates r and θ in the usual cylindrical coordinate system, as shown in the figure.

(b) [1 pt] Let ℓ be the angular momentum about the z-axis. Find the effective potential $V(r)$. Show that the turning points of the motion in r can be found by solving a cubic equation in r. (Note: You don't need to find the roots of this cubic equation.)

(c) [2 pt] Find the condition under which the particle with given ℓ makes circular motion at $r = r_0 =$ constant. Is this motion stable? If so, what is the period of small oscillations (in the variable r) about this circular motion?



- (a) $mr^2\dot{\theta} = \ell$, and $\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g\sin\alpha\cos\alpha = 0$ (discussed in the class or in Example 7.4)
- (b) The turning points can be found by solving $V(r) = \frac{\ell^2}{2mr^2} + mgr\cot\alpha = E$.
- (c) With $r_0\dot{\theta}^2\sin^2\alpha = \frac{\ell^2}{m^2 r_0^3}\sin^2\alpha = g\sin\alpha\cos\alpha$ and using an approximation similar to what we did in Eq.(8.86) or (8.99) $\rightarrow$ for the small perturbation $r \rightarrow r_0 + x$, the equation of motion found in (a) becomes $\ddot{x} + \frac{3\ell^2\sin^2\alpha}{m^2 r_0^4}x = 0$ $\rightarrow$ therefore, $\omega_{\rm osc} = \frac{\sqrt{3}\ell\sin\alpha}{mr_0^2}$ (discussed in the class or in Example 8.7)