Classical Mechanics I (Spring 2020): Midterm Exam Solution

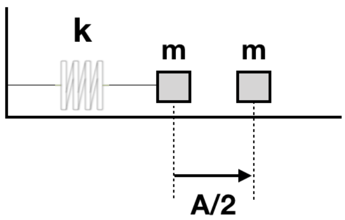
May 2, 2020

[total 15 pts, closed book/cellphone, no calculator, 90 minutes]

1. [3 pt] Consider a simple harmonic oscillator of mass m oscillating on a spring with spring constant k. The amplitude of oscillation is A. Then, at the moment the mass is at position $x = \frac{A}{2}$ moving to the right, it collides and sticks to another mass m (we call this moment $t = 0$; x is the displacement from the equilibrium point of the old oscillation).

(a) [2 pt] From momentum conservation, the speed of the resulting mass $2m$ right after the collision is half the speed of the moving mass m right before the collision. Describe the new oscillation in terms of the displacement $x(t)$ for $t > 0$. In particular, what is the amplitude of the new oscillation?

(b) [1 pt] Verify the amplitude of the new oscillation from (a) using energy conservation.



• (a) Let v_{old} be the velocity of m right before the collision. From energy conservation $\frac{1}{2}k\left(\frac{A}{2}\right)^2 + \frac{1}{2}mv_{\text{old}}^2 = \frac{1}{2}kA^2 \;\rightarrow\; v_{\text{old}} = +\frac{\sqrt{3}}{2}\sqrt{\frac{k}{m}}A$. Inserting $x(0) = \frac{A}{2}$ and $\dot{x}(0) = \frac{1}{2}v_{\text{old}} = \frac{\sqrt{3}}{4}\sqrt{\frac{k}{m}}A$ into $x(t) = A'\cos(\sqrt{\frac{k}{2m}}t + \phi)$ gives $A' = \sqrt{\frac{5}{8}}A$ and $\phi = -\tan^{-1}\sqrt{\frac{3}{2}}$.

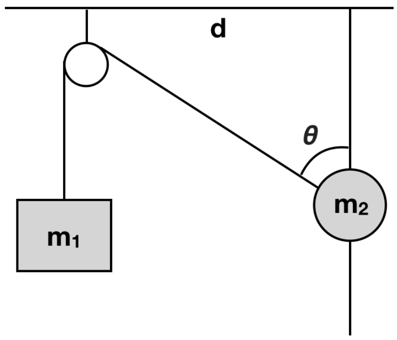
• (b) The amplitude $A' = \sqrt{\frac{5}{8}}A$ can also be acquired by energy conservation after the collision, $\frac{1}{2}k\left(\frac{A}{2}\right)^2 + \frac{1}{2}\cdot 2m\left(\frac{1}{2}v_{\text{old}}\right)^2 = \frac{1}{2}kA'^2$.

2. [3 pt] Consider the system of a pulley and two masses illustrated below. A massless string of length b is attached to a block of mass m_1, runs over a massless, frictionless pulley, then attached to a metal ball of mass m_2. The ball, with a hole through it, is threaded on a frictionless vertical rod. The rod and the pulley are separated by d. Assume that the sizes of the pulley and the ball are negligible.

(a) [1 pt] Using the variable θ shown below, find the potential energy of the system, $U(\theta)$.

(b) [1 pt] Find the equilibrium point(s). What condition should m_1 and m_2 meet for the equilibrium to occur?

(c) [1 pt] If equilibrium points do exist, determine their stabilities.



- (a) $U(\theta) = m_1 gd/\sin\theta - m_2 gd/\tan\theta + C$
- (b) For $\theta_0 = \cos^{-1}(m_2/m_1)$ to be realistic, $m_1 > m_2$.
- (c) $[d^2U/d\theta^2]_{\theta=\theta_0} = \left[(m_1 \sin^2\theta - 2\cos\theta(m_2 - m_1\cos\theta)) \cdot \frac{gd}{\sin^3\theta}\right]_{\theta=\theta_0} = \frac{m_1 gd}{\sin\theta_0} > 0$

3. [2 pt] In the class we covered several special topics including tidal force, dark matter, and (super)massive black holes.

(a) [1 pt] We discussed several evidences that suggest the existence of invisible entities such as dark matter and (super)massive black holes. Describe one (or more) of them per each of the entities below. 2-3 sentences per evidence are expected to clearly explain how each evidence points to the existence of the invisible entity. Use diagrams if desired.

- Evidence for the existence of dark matter
- Evidence for the existence of (super)massive black holes

(b) [1 pt] We also discussed how the tidal force manifests itself in various settings. Describe one (or more) exemplary phenomenon per each of the categories below. 2-3 sentences per phenomenon are expected to clearly explain how the concept of tidal force is used. Use diagrams if desired.

- Tidal interaction on satellite moons
- Tidal interaction on galaxies
- Tidal interaction on objects approaching very closely to a massive body.

• (a-1) Flat galactic rotation curve, fast random motion of member galaxies in galaxy clusters, gravitational lensing, dissociation of the gravitational potential and baryons during collisions of galaxy clusters (e.g., “bullet cluster”), cosmological structure formation.

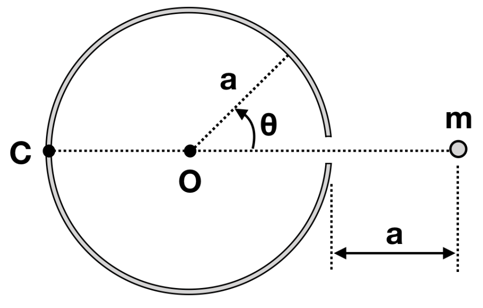
• (a-2) Fast motion of stars near the center of Milky Way galaxy, gravitational redshift measurement near the center of Milky Way galaxy, direct imaging of the shadow of a massive black hole at the center of M87 galaxy. For more information about each of the items above, see the class slides, Lectures 3-1 and 8-2.

• (b-1) Tidal interaction on satellite moons: ocean tides, spring and neap tides, tidal locking of the Earth+Moon system, tidal locking of the Pluto+Charon system, slowdown of Earth's rotation, tidal locking of the Jupiter+Io system, tidal heating on Io and Europa by Jupiter, tidal heating on Enceladus by Saturn

• (b-2) Tidal interaction on galaxies: tidal stripping or harassment of satellite galaxies, tidal tails or bridges during galaxy collision

• (b-3) Tidal interaction on objects approaching very closely to a massive body: tidal disintegration of Comet Shoemaker-Levy when approaching Jupiter, tidal disruption of a star or a gas cloud when approaching a black hole, spaghettification of an object approaching a black hole. For more information about each of the items above, see the class slide, Lecture 9-1.

4. [2 pt] Consider a thin, uniform spherical shell of mass M and radius a with a very small opening. A small bead of mass m and negligible size is released from a distance a in front of the opening. Calculate the speed with which the bead hits the point C on the shell, opposite to the opening. In solving this problem, you are asked to *directly* compute the gravitational field vector $\mathbf{g}(r)$ inside and outside the shell as a function of distance r from the center of the shell, O (that is, not by using the gravitational potential or by utilizing Poisson's equation). $\hat{\mathbf{e}}_r$ is the unit vector along r pointing away from O. Consider only gravitational force, and assume no other gravitational source.



• Starting from Eq.(5.46) with $a^2 = x^2 + r^2 - 2xr\,\cos\alpha$,

$$|\mathbf{g}(r)| = G\int_V \frac{\rho(\mathbf{r}')dv'}{|\mathbf{r}-\mathbf{r}'|^2}\cos\alpha = 2\pi\rho_s G\int_0^{\pi}\frac{a^2\sin\theta\,\cos\alpha\,d\theta}{x^2} = \frac{\pi\rho_s aG}{r^2}\int_{|r-a|}^{r+a}\frac{x^2+r^2-a^2}{x^2}dx$$

$$\rightarrow\ \mathbf{g}(r>a) = -\frac{\pi\rho_s aG}{r^2}(4a)\,\hat{\mathbf{e}}_r = -\frac{GM}{r^2}\,\hat{\mathbf{e}}_r\,,\ \text{and}\ \mathbf{g}(r<a) = -\frac{\pi\rho_s aG}{r^2}(0)\,\hat{\mathbf{e}}_r = 0$$

• Thus, the bead keeps its kinetic energy when it passes the opening, $\frac{1}{2}mv^2 = -\frac{GMm}{2a} + \frac{GMm}{a}$, until it hits the point C.

5. [3 pt] Consider a thin rod (line mass) of uniform line density λ. Consider only gravitational force, and assume no other gravitational source.

(a) [1 pt] If the rod is infinitely long, find the gravitational field vector $\mathbf{g}(r)$ at distance r from the rod in a direction perpendicular to the rod by using the integral form of Poisson's equation. $\hat{\mathbf{e}}_r$ is the unit vector along r pointing away from the rod.

(b) [2 pt] If the rod has a finite length of 2ℓ, determine the gravitational field vector $\mathbf{g}(r)$ at distance r from the rod's midpoint O in a direction perpendicular to the rod. Show that your answer becomes that of (a) when ℓ approaches ∞. One may use the following formula from an integral table: $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2 + a^2})$.

• (a) From Eq.(5.35), one gets

$$\int_S \mathbf{n} \cdot \mathbf{g} da = -4\pi G \int_V \rho(\mathbf{r}') dv' \;\rightarrow\; 2\pi r L\, \hat{\mathbf{e}}_r \cdot \mathbf{g}(r) = -4\pi G\lambda L \;\rightarrow\; \mathbf{g}(r) = -\frac{2G\lambda}{r}\hat{\mathbf{e}}_r$$

• (b-1) From Eq.(5.7), one gets

$$\Phi(r) = -G\int_{-\ell}^{+\ell} \frac{\lambda\, dz}{\sqrt{z^2+r^2}} = -G\lambda \ln\left(\frac{\sqrt{\ell^2+r^2}+\ell}{\sqrt{\ell^2+r^2}-\ell}\right)$$

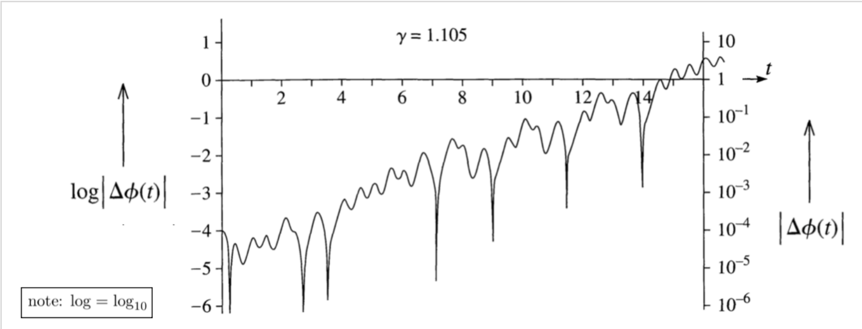
$$\rightarrow\; g(r) = -\frac{\partial\Phi}{\partial r} = G\lambda\left[\frac{r/\sqrt{\ell^2+r^2}}{\sqrt{\ell^2+r^2}+\ell} - \frac{r/\sqrt{\ell^2+r^2}}{\sqrt{\ell^2+r^2}-\ell}\right] = -\frac{2G\lambda}{r\sqrt{1+(\frac{r}{l})^2}} \xrightarrow[\ell\to\infty]{} -\frac{2G\lambda}{r}$$

• (b-2) Or, when ℓ approaches ∞,

$$\Phi(r) \simeq -G\lambda \ln\left(\frac{1+\frac{r^2}{2\ell^2}+1}{1+\frac{r^2}{2\ell^2}-1}\right) \simeq -G\lambda\left[\ln\left(1+\frac{r^2}{4\ell^2}\right) - \ln\left(\frac{r^2}{4\ell^2}\right)\right] \simeq G\lambda\ln\left(\frac{r^2}{4\ell^2}\right)$$

$$= 2G\lambda\ln(r) + C' \;\rightarrow\; \mathbf{g}(r) = -\frac{2G\lambda}{r}\hat{\mathbf{e}}_r$$

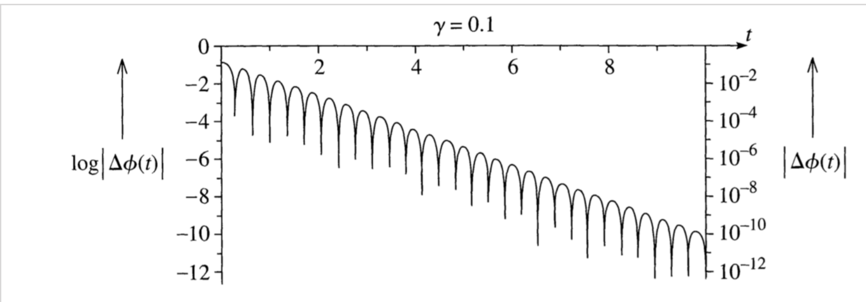
6. [2 pt] Consider a damped driven plane pendulum described by $\ddot{\phi}+2\beta\dot{\phi}+\omega_0^2\sin\phi = \gamma\omega_0^2\cos\omega t$, where ϕ is the angular displacement in radian at time t, ω is the driving frequency, $\omega_0 = 1.5\,\omega$ is the natural frequency of the pendulum, and $\beta = 3\,\omega/8$ is the damping constant. For simplicity, we choose the driving frequency to be $\omega = 2\pi$ so that the drive cycle τ becomes 1. Now let us consider two identical pendulums that satisfy the exact same equation of motion, but have slightly different initial conditions. We denote the separation of these two pendulums as $\Delta\phi(t) = \phi_2(t) - \phi_1(t)$. For $\gamma = 1.105$, $|\Delta\phi(t)|$ starts out at 10^{-4} but reaches π by $t = 16$, as seen in the figure below.



(a) [1 pt] Why can $t = 16$ be considered an important milestone? Use this observation to estimate the Lyapunov exponent λ, as defined in $\Delta\phi(t) \sim |\Delta\phi(0)|e^{\lambda t}$. Then, suppose that you want to predict the pendulum's $\phi(t)$ with an accuracy of 10^{-2} and that you know the initial value $\phi(0)$ within 10^{-6}. Estimate the maximum time $T_{\max}$ for which you can predict $\phi(t)$ within the required accuracy. (Note: Here "$\sim$" signifies that $\Delta\phi(t)$ on average oscillates roughly underneath the envelope $|\Delta\phi(0)|e^{\lambda t}$. Obtain your answers accurate to only two significant figures. Use the following if necessary: $\log(\pi) = \log_{10}(\pi) \approx 0.50$, $\log(e) = \log_{10}(e) \approx 0.43$.)

(b) [0.5 pt] How would your answer $T_{\max}$ change in (a) if you improved the accuracy of your measurement of the initial value to 10^{-8} (a hundred-fold improvement with a vast investment of money and resources)? Use this example to explain the difficulty in making accurate long-term predictions for chaotic systems.

(c) [0.5 pt] Compare qualitatively the two cases, $\gamma = 1.105$ (figure above) and $\gamma = 0.1$ (figure below), focusing on the evolution of $|\Delta\phi(t)|$. Include and *circle* the following keywords in your answer: exponential decay and/or growth, sensitivity to initial conditions, linear, chaotic. (Note: You don't need to solve differential equations here; qualitative justifications are more than enough.)



• (a) An uncertainty of $\pm\pi$ radian in ϕ means that we have absolutely no way of predicting where the pendulum is. $\log|\Delta\phi(t)| \approx \log|\Delta\phi(0)| + \lambda t \cdot \log(e) \;\rightarrow\; \log(\pi) \approx \log(10^{-4}) + 16\lambda \cdot \log(e) \;\rightarrow$ Thus, $0.50 \approx -4 + 6.9\lambda$ gives $\lambda \approx 0.65$, confirming $\lambda > 0$ for chaotic behavior. Then, $\log(10^{-2}) \approx \log(10^{-6}) + 0.65\,T_{\max} \cdot \log(e) \;\rightarrow\; T_{\max} \approx 14$.

• (b) $\log(10^{-2}) \approx \log(10^{-8}) + 0.65\,T_{\max} \cdot \log(e) \;\rightarrow\; T_{\max} \approx 21$, only $\sim 50\%$ increase in $T_{\max}$ with a hundred-fold improvement in accuracy of the initial value measurement.

• (c) In the case of $\gamma = 0.1$, on average $|\Delta\phi(t)|$ decays exponentially demonstrating that the motion becomes insensitive to its initial condition ("linear" behavior). In contrast, in the case of $\gamma = 1.105$, on average $|\Delta\phi(t)|$ grows exponentially demonstrating that the motion becomes very sensitive to its initial condition ("chaotic" behavior).