Classical Mechanics I (Spring 2020): Final Examination

Jun. 20, 2020

[total 25 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from (1) to (6) on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet (2)). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

• Make your writing easy to read. Illegible answers will *not* be graded. When asked, circle the keywords in your answer clearly.

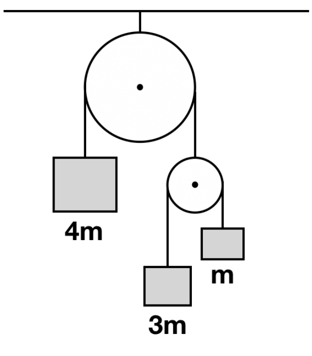
1. [4 pt] Consider a double pulley system shown below with massless inextensible strings and two massless smooth pulleys. The masses of the three weights are indicated in the figure (m, $3m$, $4m$).

(a) [1 pt] Determine the Lagrangian and Lagrange's equations of motion (without using undetermined multipliers). You will first need to define your choice of the generalized coordinates.

(b) [1 pt] Determine the downward acceleration of each of the three masses.

(c) [1 pt] Repeat (a) using the Lagrangian method with undetermined multipliers. Identify the forces of constraint and explain their physical meanings.

(d) [1 pt] Repeat (a) using the Newtonian method.



2. [4 pt] Show that the shortest path between two given points on the surface of a sphere is a *great circle* – in three different ways below.

(a) [2 pt] By introducing a functional in the form of $\int f_1\{\theta, \theta'\}\, d\phi$ in spherical coordinates $[r, \theta, \phi]$. One may opt to use the formula $\int \frac{a\csc^2\theta d\theta}{\sqrt{1-a^2\csc^2\theta}} = \sin^{-1}\left(\frac{\cot\theta}{\sqrt{(1/a^2)-1}}\right) + \text{constant}$ (when $a < 1$) to expedite the derivation.

(b) [1 pt] By using an alternative functional form of $\int f_2\{\theta, \phi'\}\, d\theta$.

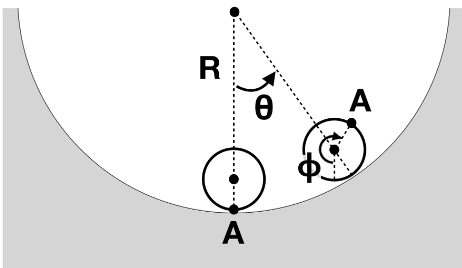
(c) [1 pt] By rewriting the functional in the form of $\int f_3\{y', z'\}\, dx$ in Cartesian coordinates $[x, y, z]$, and employing the method of Euler equations with an auxiliary condition imposed.

3. [5 pt] Consider a homogenous disk of radius ρ and mass m, which rolls without slipping inside the lower half of a fixed, hollow cylinder of inner radius R. As indicated in the figure below, θ is the angle between the vertical and the line joining the centers of the disk and the cylinder, and ϕ is the angle by which the disk rotated from the bottom of the cylinder.

(a) [2 pt] Find the period of small oscillations in the variable θ using the Lagrangian method (without undetermined multipliers).

(b) [1 pt] Repeat (a) using the Lagrangian method with undetermined multipliers.

(c) [2 pt] Repeat (a) using the Hamiltonian method.



4. [3 pt] Throughout the semester we discussed many examples in which concepts in classical mechanics are utilized in contemporary research and in explaining daily phenomena.

(a) [2 pt] We discussed how invisible objects in the Universe were discovered by modeling a two-body system interacting via the central force. Describe three or more such cases. 2-3 sentences per case are expected to clearly explain how people came to notice the existence (and mass) of the invisible object using the concept of two-body motion, central force, and/or reduced mass. Use diagrams if desired.

(b) [1 pt] In the last class of the semester, five of your peers presented their term projects. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 2-3 sentences is expected to clearly convey the core physics idea of his/her term project. If you were one of the presenters, please choose someone else's.

5. [4 pt] Consider a particle of mass μ moving under the influence of an attractive central force of the form $F(r) = -\frac{k}{r^2}\exp\left(-\frac{r}{a}\right)$ where r is the distance from the origin of the central force, and k and a are positive constants.

(a) [2 pt] By using the concept of an effective potential, find the condition for radius r_0 at which the particle with angular momentum ℓ moves in a circular orbit.

(b) [2 pt] Then, again using the concept of the effective potential, show that, if $r_0 < a$, a slight radial nudge to this circular orbit causes only small radial oscillations. Find the period for these oscillations.

6. [5 pt] A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α (see the figure below). The particle is subject to the gravitational field g.

(a) [2 pt] Determine the Lagrangian and Lagrange's equations of motion for the coordinates r and θ in the usual cylindrical coordinate system, as shown in the figure.

(b) [1 pt] Let ℓ be the angular momentum about the z-axis. Find the effective potential $V(r)$. Show that the turning points of the motion in r can be found by solving a cubic equation in r. (Note: You don't need to find the roots of this cubic equation.)

(c) [2 pt] Find the condition under which the particle with given ℓ makes circular motion at $r = r_0 =$ constant. Is this motion stable? If so, what is the period of small oscillations (in the variable r) about this circular motion?