Classical Mechanics I (Spring 2020): Midterm Examination

May 2, 2020

[total 15 pts, closed book/cellphone, no calculator, 90 minutes]

• By 21:00pm, you are asked to (1) sign the Honor Code, (2) work on the following questions, (3) scan your answers, and (4) turn in your answers to eTL as a single pdf file (eTL does not accept multiple files). You have to stay in the meeting room and be visible to the TAs for the whole 90 minutes even if you have nothing to write down. Have your cellphone and student ID nearby. Prepare to pick up the phone and follow the TA's instructions (e.g., show your ID to the camera).

• It is *strongly* advised that you start to work on scanning and uploading your answer sheets by no later than 20:50pm. If you submit your work after 21:00pm, your score will be reduced by 2 point for each minute that the submission is late. If, for some reason, you experience technical difficulties on eTL, email your answer as a single pdf file to the lecturer immediately. The time the email is *received* will be marked as your submission time.

• First, bring out a piece of paper and make a cover page (first page) of your answer sheets. Copy the following in your own handwriting and sign. Submit this as part of your answer.

명예규율 (Honor Code) 서약서

교과목명: 역학 1
담당교수: 김지훈
시험일: 2020년 5월 2일

나 (   )은/는 위 교과의 시험시간 중 어떠한 종류의 부정행위도 하지 않았음을 서약합니다.

소속 (학부/과):
학번:
이름:
서명 (혹은 도장):

• Then, make sure to prepare *at least* 6 pieces of A4 papers for your answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from 1 to 6 on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet 2). If you need additional papers, write down the number clearly so the grader can locate your answers without difficulty. After the exam, you will turn in all 6 answer sheets, even if some are still blank.

• Your original answer sheets should be on physical A4 papers. In other words, do not make your answer sheets digitally using a tablet/iPad and an electronic pencil. Keep your original answer sheets on papers for your record. You will be asked to submit the original when we switch to in-class lectures.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you have any issue or question on the problem itself or on English expressions, you *must* raise it to your microphone in the first 30 minutes.

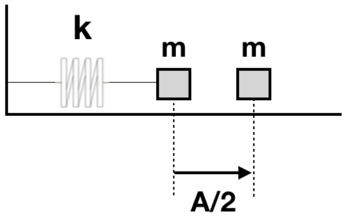
• Make your writing easy to read, and double check your scanned answers before submitting it to eTL. Illegible answers will *not* be graded.

• Do *not* use the Lagrangian method.

1. [3 pt] Consider a simple harmonic oscillator of mass m oscillating on a spring with spring constant k. The amplitude of oscillation is A. Then, at the moment the mass is at position $x = \frac{A}{2}$ moving to the right, it collides and sticks to another mass m (we call this moment $t = 0$; x is the displacement from the equilibrium point of the old oscillation).

(a) [2 pt] From momentum conservation, the speed of the resulting mass $2m$ right after the collision is half the speed of the moving mass m right before the collision. Describe the new oscillation in terms of the displacement $x(t)$ for $t > 0$. In particular, what is the amplitude of the new oscillation?

(b) [1 pt] Verify the amplitude of the new oscillation from (a) using energy conservation.

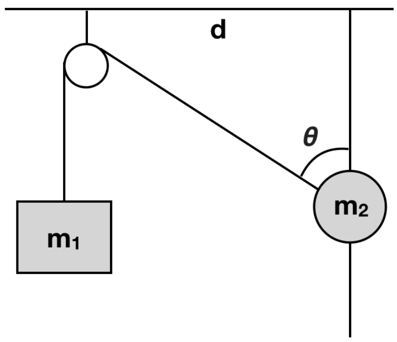


2. [3 pt] Consider the system of a pulley and two masses illustrated below. A massless string of length b is attached to a block of mass m_1, runs over a massless, frictionless pulley, then attached to a metal ball of mass m_2. The ball, with a hole through it, is threaded on a frictionless vertical rod. The rod and the pulley are separated by d. Assume that the sizes of the pulley and the ball are negligible.

(a) [1 pt] Using the variable θ shown below, find the potential energy of the system, $U(\theta)$.

(b) [1 pt] Find the equilibrium point(s). What condition should m_1 and m_2 meet for the equilibrium to occur?

(c) [1 pt] If equilibrium points do exist, determine their stabilities.



3. [2 pt] In the class we covered several special topics including tidal force, dark matter, and (super)massive black holes.

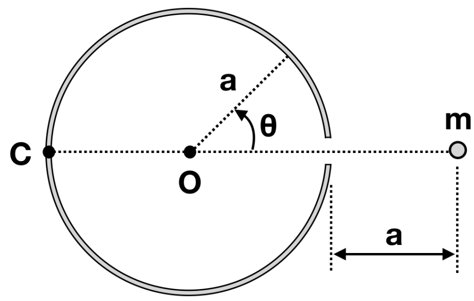
(a) [1 pt] We discussed several evidences that suggest the existence of invisible entities such as dark matter and (super)massive black holes. Describe one (or more) of them per each of the entities below. 2-3 sentences per evidence are expected to clearly explain how each evidence points to the existence of the invisible entity. Use diagrams if desired.

- Evidence for the existence of dark matter
- Evidence for the existence of (super)massive black holes

(b) [1 pt] We also discussed how the tidal force manifests itself in various settings. Describe one (or more) exemplary phenomenon per each of the categories below. 2-3 sentences per phenomenon are expected to clearly explain how the concept of tidal force is used. Use diagrams if desired.

- Tidal interaction on satellite moons
- Tidal interaction on galaxies
- Tidal interaction on objects approaching very closely to a massive body.

4. [2 pt] Consider a thin, uniform spherical shell of mass M and radius a with a very small opening. A small bead of mass m and negligible size is released from a distance a in front of the opening. Calculate the speed with which the bead hits the point C on the shell, opposite to the opening. In solving this problem, you are asked to *directly* compute the gravitational field vector $\mathbf{g}(r)$ inside and outside the shell as a function of distance r from the center of the shell, O (that is, not by using the gravitational potential or by utilizing Poisson's equation). $\hat{\mathbf{e}}_r$ is the unit vector along r pointing away from O. Consider only gravitational force, and assume no other gravitational source.

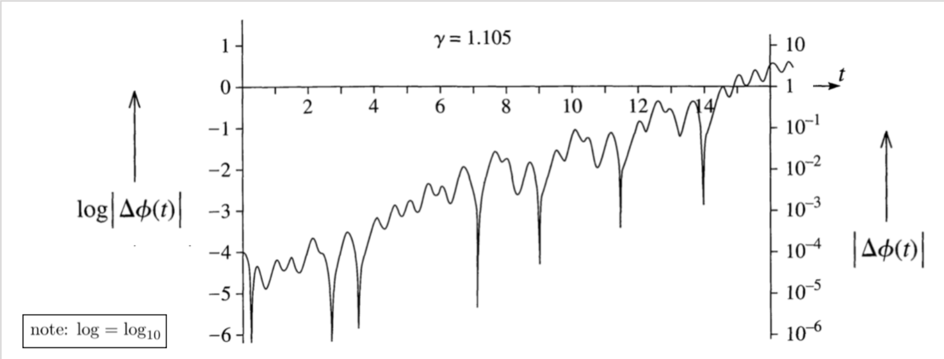


5. [3 pt] Consider a thin rod (line mass) of uniform line density λ. Consider only gravitational force, and assume no other gravitational source.

(a) [1 pt] If the rod is infinitely long, find the gravitational field vector $\mathbf{g}(r)$ at distance r from the rod in a direction perpendicular to the rod by using the integral form of Poisson's equation. $\hat{\mathbf{e}}_r$ is the unit vector along r pointing away from the rod.

(b) [2 pt] If the rod has a finite length of 2ℓ, determine the gravitational field vector $\mathbf{g}(r)$ at distance r from the rod's midpoint O in a direction perpendicular to the rod. Show that your answer becomes that of (a) when ℓ approaches ∞. One may use the following formula from an integral table: $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2 + a^2})$.

6. [2 pt] Consider a damped driven plane pendulum described by $\ddot{\phi}+2\beta\dot{\phi}+\omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t$, where ϕ is the angular displacement in radian at time t, ω is the driving frequency, $\omega_0 = 1.5\,\omega$ is the natural frequency of the pendulum, and $\beta = 3\,\omega/8$ is the damping constant. For simplicity, we choose the driving frequency to be $\omega = 2\pi$ so that the drive cycle τ becomes 1. Now let us consider two identical pendulums that satisfy the exact same equation of motion, but have slightly different initial conditions. We denote the separation of these two pendulums as $\Delta\phi(t) = \phi_2(t) - \phi_1(t)$. For $\gamma = 1.105$, $|\Delta\phi(t)|$ starts out at 10^{-4} but reaches π by $t = 16$, as seen in the figure below.



(a) [1 pt] Why can $t = 16$ be considered an important milestone? Use this observation to estimate the Lyapunov exponent λ, as defined in $\Delta\phi(t) \sim |\Delta\phi(0)|e^{\lambda t}$. Then, suppose that you want to predict the pendulum's $\phi(t)$ with an accuracy of 10^{-2} and that you know the initial value $\phi(0)$ within 10^{-6}. Estimate the maximum time $T_{\max}$ for which you can predict $\phi(t)$ within the required accuracy. (Note: Here "$\sim$" signifies that $\Delta\phi(t)$ on average oscillates roughly underneath the envelope $|\Delta\phi(0)|e^{\lambda t}$. Obtain your answers accurate to only two significant figures. Use the following if necessary: $\log(\pi) = \log_{10}(\pi) \approx 0.50$, $\log(e) = \log_{10}(e) \approx 0.43$.)

(b) [0.5 pt] How would your answer $T_{\max}$ change in (a) if you improved the accuracy of your measurement of the initial value to 10^{-8} (a hundred-fold improvement with a vast investment of money and resources)? Use this example to explain the difficulty in making accurate long-term predictions for chaotic systems.

(c) [0.5 pt] Compare qualitatively the two cases, $\gamma = 1.105$ (figure above) and $\gamma = 0.1$ (figure below), focusing on the evolution of $|\Delta\phi(t)|$. Include and *circle* the following keywords in your answer: exponential decay and/or growth, sensitivity to initial conditions, linear, chaotic. (Note: You don't need to solve differential equations here; qualitative justifications are more than enough.)