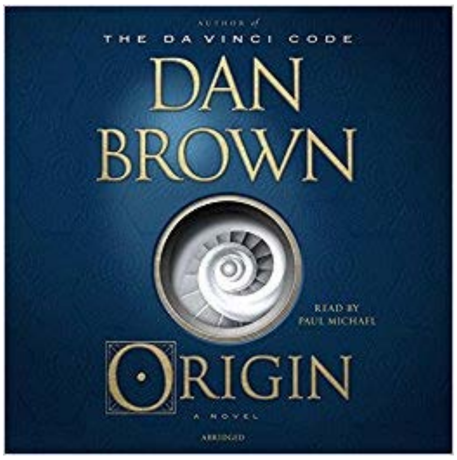


To spread energy.

In the simplest terms, when the universe found areas of *focused* energy, it spread that energy out. The classic example, as Kirsch had mentioned, was the cup of hot coffee on the counter; it always cooled, dispersing its heat to the other molecules in the room in accordance with the Second Law of Thermodynamics.

Infinitely Many Solutions to the Black-Scholes PDE; Physics Point of View

서울대학교 물리학과

2018. 05. 23. 16:00 (56동 106호)

최병선(경제학부) 최무영(물리천문학부)

Featuring:
최병선 Pictures

CBS@The Wave
April, 2018

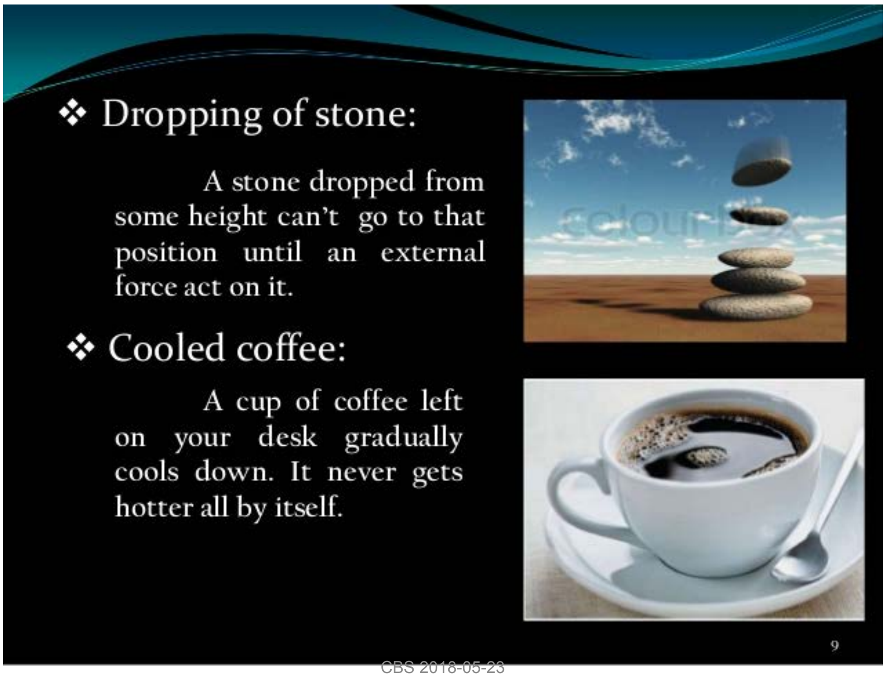
Agenda (previous)

- A General Solution to Black-Scholes Boundary Value Problem (Business)
- Derivation (Mathematics)
- Central Limit Theorem (Statistics)

Agenda (Today)

- Appropriateness of Current Solution of Heat Transfer/Diffusion Boundary Value Problem
- Appropriateness of Black-Scholes Formula
- Completeness of Hermite Polynomials
- Jumps in a Solution of Diffusion Equation
- Continuity of (Binary) Option Price
- Appropriateness of Feynman-Kac Formula
- Monte Carlo Simulation for Dynamical System
- Risk Neutral Option Pricing
- A Minimum Condition of Stochastic Calculus

The Second Law of Thermodynamics



$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

Diffusion (aka Heat Transfer) Equation

- Consider a boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \quad \text{with } u(0,t) = A, \ (t > 0)$$

- The traditional/current solution is

$$u^0(x,t) = A \operatorname{erfc}\left(\frac{x}{\sqrt{2t}}\right).$$

A Generalized Solution

- Kang, Kim, Choi, and Choi (KKCC, K^2C^2, 2018)

$$u^\zeta(x,t) = u^0(x,t) + \sum_{m=0}^{M} \zeta_{2m+1} \frac{1}{t^{(2m+1)/2}} He_{2m+1}\left(\frac{x}{\sqrt{t}}\right) \phi\left(\frac{x}{\sqrt{t}}\right),$$

where M is any non-negative integer, $\zeta_1, \zeta_3, \cdots$ are any numbers, $He_n(x)$ is probabilists' Hermite polynomial of order n, and

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right).$$

Probabilists' Hermite Polynomials

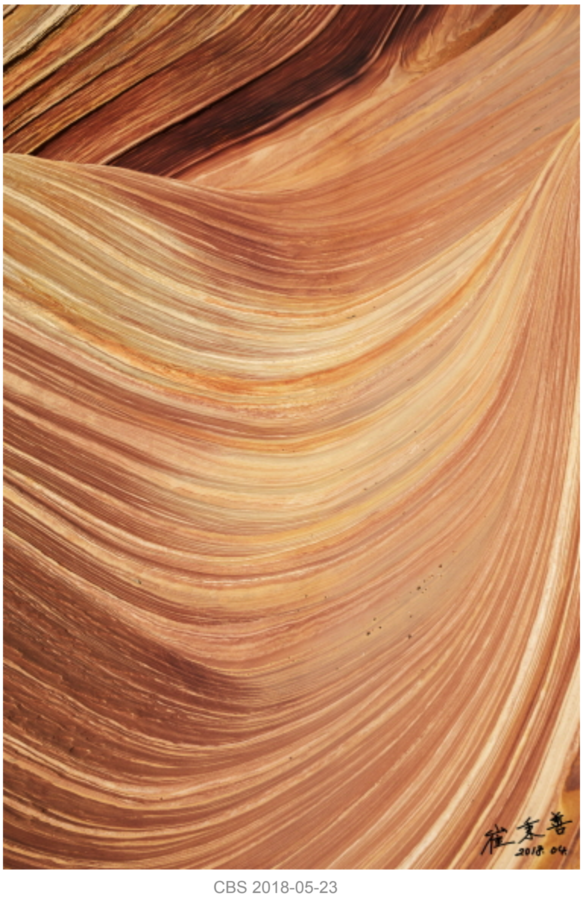
$$He_n(x) \equiv (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1.$$

$$He_0(x) = 1$$
$$He_1(x) = x$$
$$He_2(x) = x^2 - 1$$
$$He_3(x) = x^3 - 3x$$
$$He_4(x) = x^4 - 6x^2 + 3$$
$$He_5(x) = x^5 - 10x^3 + 15x$$
$$He_6(x) = x^6 - 15x^4 + 45x^2 - 15$$

K^2C^2 Examples for the Generalized Solution

(1) Schrödinger Equation
(2) Drift-Diffusion in Positive-Negative
Junction Pixel Sensors
(3) Higuchi Equation for Drug Release
(4) Black-Scholes PDE for a Call Option

Some results will be presented near future by M. Choi, H. Kang, and C. Kim.



Black-Scholes Environments

- Black-Scholes (1973) Assumptions
- Underlying x_t w/ $Var\left(\Delta \ln\left(x_{t+\Delta t}/x_t\right)\right) = v^2 \Delta t$
- European call option $w(x_T, T) = [x_T - K]^+$
- Strike K at expiry T
- No transaction costs
- Can borrow any fraction of a security
- No penalties to short selling
- Constant short term interest rate r

Black-Scholes Boundary Value Problem

- Black-Scholes PDE

$$w_t(x,t) = rw(x,t) - rxw_x(x,t) - \frac{1}{2}v^2x^2w_{xx}(x,t)$$

- Boundary condition (Terminal condition)

$$\lim_{t \uparrow T} w(x,t) = [x - K]^+, \quad (x \neq K)$$

Black-Scholes Formula (1973)

- There exists the UNIQUE solution

$$w^{BS}(x_t, t) = x_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

- $\tau \equiv T - t$

$$d_1 \equiv \frac{1}{v\sqrt{\tau}}\left\{\ln\frac{x_t}{K} + \left[r + \frac{v^2}{2}\right]\tau\right\}$$

$$d_2 \equiv \frac{1}{v\sqrt{\tau}}\left\{\ln\frac{x_t}{K} + \left[r - \frac{v^2}{2}\right]\tau\right\}$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(z)dz$$

- 1997 Nobel Prize in Economic Sciences

Choi-Choi Formula (2018)

- $$w^{\zeta}(x_t, t) \equiv w^{BS}(x_t, t) + \sum_{l=0}^{M} \zeta_l C_l(x_t, t)$$

  with

  $$C_l(x_t, t) \equiv K e^{-r\tau} \phi(d_2) \frac{1}{(v\sqrt{\tau})^{l+1}} He_l \left(\frac{1}{v\sqrt{\tau}} \ln \frac{x_t}{K} \right)$$

  for any $\quad M \in Z_{\geq 0}, \quad \zeta = (\zeta_1, \zeta_1, \cdots, \zeta_M) \in R^M.$

- Against the law of one price

Solution Check

```
When order = 61, the Echeck value has 0.
When order = 62, the Echeck value has 0.
When order = 63, the Echeck value has 0.
When order = 64, the Echeck value has 0.
When order = 65, the Echeck value has 0.
When order = 66, the Echeck value has 0.
When order = 67, the Echeck value has 0.
When order = 68, the Echeck value has 0.
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When order = 85, the Echeck value has 0.
When order = 86, the Echeck value has 0.
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When order = 90, the Echeck value has 0.
When order = 91, the Echeck value has 0.
When order = 92, the Echeck value has 0.
When order = 93, the Echeck value has 0.
When order = 94, the Echeck value has 0.
When order = 95, the Echeck value has 0.
When order = 96, the Echeck value has 0.
When order = 97, the Echeck value has 0.
When order = 98, the Echeck value has 0.
When order = 99, the Echeck value has 0.
When order = 100, the Echeck value has 0.
```
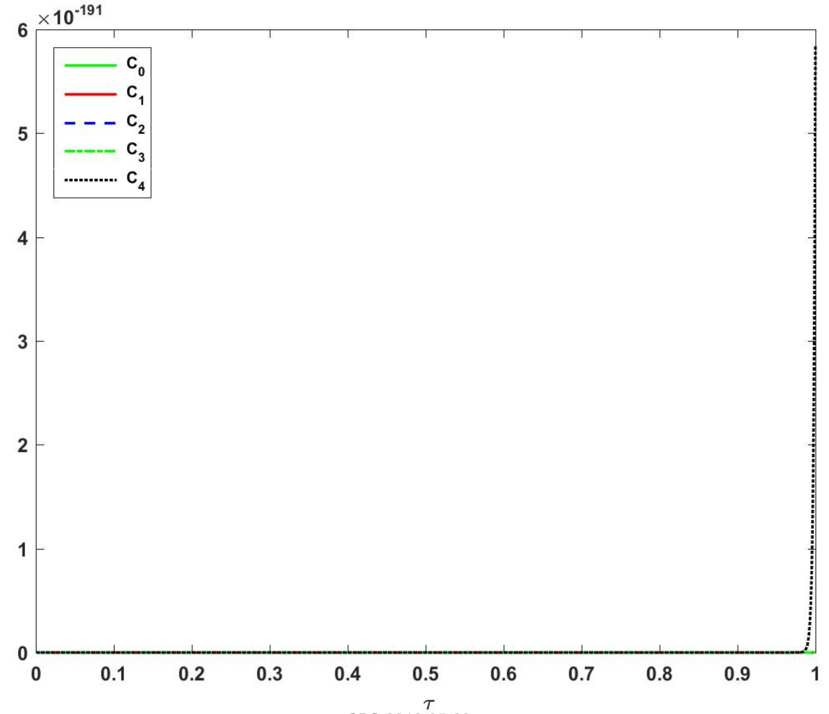
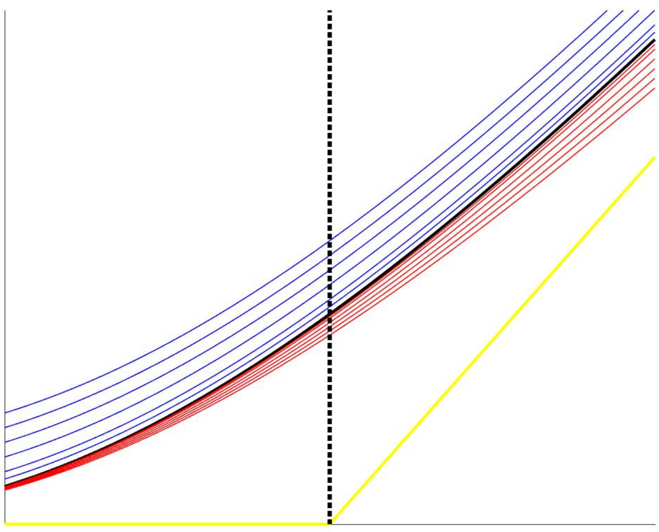

$$K = 100,\ x_t = 120,\ r = 0.05,\ v = 0.03$$



Completeness of Hermite Polynomials

- Hermite polynomials $\{He_n(x)\,;\, n = 0, 1, \cdots\}$ form an orthogonal basis of $L^2(R)$ of functions satisfying $\int_{-\infty}^{\infty} |f(x)|^2 \phi(x) dx < \infty.$
- An orthogonal basis for $L^2(R)$ is complete.

Completeness of $\{He_k(x)\}$



Intermediate Summary

The fact remains, however, that the B&S paper is the decisive breakthrough in the subject. Any history of option pricing—or of financial economics generally—divides in black-and-white terms into the pre-Black–Scholes and post-Black–Scholes eras.

Louis Bachelier's *Theory of Speculation*
THE ORIGINS OF MODERN FINANCE
Translated and with Commentary by
Mark Davis and *Alison Etheridge*

Do we need the Black-Scholes Partial Differential Equation?

No! Practically we need only the Terminal Condition.

Moreover, we should reconsider

- Exotic Option Prices; Barrier options, Lookback options, Asian options, Spread options, … ,
- Stochastic Volatility; Heston model, Chen model, …
- Interest-Rate Moddel; Vasicek model, Hull-White model, Cox-Ingersoll-Ross model, Longstaff-Schwartz model, … .
- Almost all the models discussed at *The Complete Guide to Option Pricing Formulas* by Haug, E.G. (2007)

FOURIER'S HEAT CONDUCTION EQUATION from Narasimhan (1999)

TABLE 1. Chronology of Significant Contributions on Diffusion

	Year	*Contribution*
Fahrenheit	1724	mercury thermometer and standardized temperature scale
Abbé Nollet	1752	observation of osmosis across animal membrane
Bernoulli	1752	use of trigonometric series for solving differential equation
Black	1760	recognition of latent heat and specific heat
Crawford	1779	correlation between respiration of animals and their body heat
Lavoisier and Laplace	1783	first calorimeter; measurement of heat capacity, latent heat
Laplace	1789	formulation of Laplace operator
Biot	1804	heat conduction among discontinuous bodies
Fourier	1807	partial differential equation for heat conduction in solids
Fourier	1822	*Théorie Analytique de la Chaleur*
Ohm	1827	law governing current flow in electrical conductors
Dutrochet	1827	discovery of endosmosis and exosmosis
Green	1828	formal definition of a potential
Graham	1833	law governing diffusion of gases
Thomson	1842	similarities between equations of heat diffusion and electrostatics
Poiseuille	1846	experimental studies on water flow through capillaries
Graham	1850	experimental studies on diffusion in liquids
Fick	1855	Fourier's model applied to diffusion in liquids
Darcy	1856	law governing flow of water in porous media
[illegible]	[illegible]	potential theory applied to flow in groundwater basins

Do we practically need the Diffusion/Heat-Transfer Partial Differential Equation?

No! Practically we need only the Boundary Conditions.

Speaking Boldly !!!!!!!

points was of no importance to begin with.

Thus the partial differential equation entered theoretical physics as a handmaid, but has gradually become mistress. This began in the nineteenth century when the wave-theory of light

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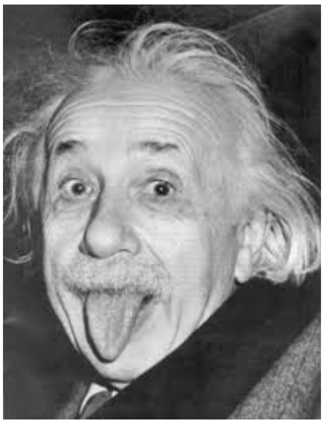


MAXWELL'S INFLUENCE ON THE DEVELOPMENT OF THE CONCEPTION OF PHYSICAL REALITY

Written for the centenary of Maxwell's birth [1931]

- ~~*The World AS I See It (p. 63)*~~

Newtonian Paradigm

- The first paradigm, which we shall refer to as the Newtonian, was established in the seventeenth century. According to this approach, a dynamical system is understood by modeling it with a differential equation and then solving that equation.
- We call this the Newtonian model without prejudice as to what Newton's world view may actually have been. It might be argued that 'Laplacian' is a more appropriate term.

- Rapp, P.E., Schmah, T.I., & Mees, A.I. (1999) Models of knowing and the investigation of dynamical systems, *Physica D* **132**, pp. 133–149.

Question about Newtonian Paradigm

https://www.researchgate.net/post/Are_differential_equations_the_proper_tool_to_describe_reality12

Question Asked 6 years ago

Marek Wojciech Gutowski

95.12 · Institute of Physics of the Polish Academy of Sciences

Are differential equations the proper tool to describe reality?
Newton introduced differential equations to physics, some 200 years ago. Later Maxwell added his own set. We also have Navier-Stokes equation, and of course - Schroedinger equation. All they were big steps in science, no doubts. But I feel uneasy, when I see, for example in thermodynamics,

differentiation with respect to the (discrete!) number of particles. That's clear abuse of a beautiful and well established mathematical concept - yet nobody complains or even raises this question. Our world seems discrete (look at STM images if you don't like XIX-th century Dalton's law), so perhaps we need some other mathematical tool(s) to describe it correctly? Maybe graph theory?

Countable Boundary Conditions

- Function $f(x)$
- Series representation (e.g., Taylor series)

$$\sum_{n=0}^{\infty} \alpha_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

- To determine $\alpha_0, \alpha_1, \cdots, \alpha_M$, we need

$$f(x_0), f(x_1), \cdots, f(x_M).$$

- To determine the coefficients $\zeta_0, \zeta_1, \cdots$ in KKCC formula, we need countably infinite number of boundary conditions.

- But we have only finite number of boundary conditions. Thus, countably infinite number of free coefficients for a solution of the boundary value problem, which means the solution space of the boundary value problem is of infinite dimension.

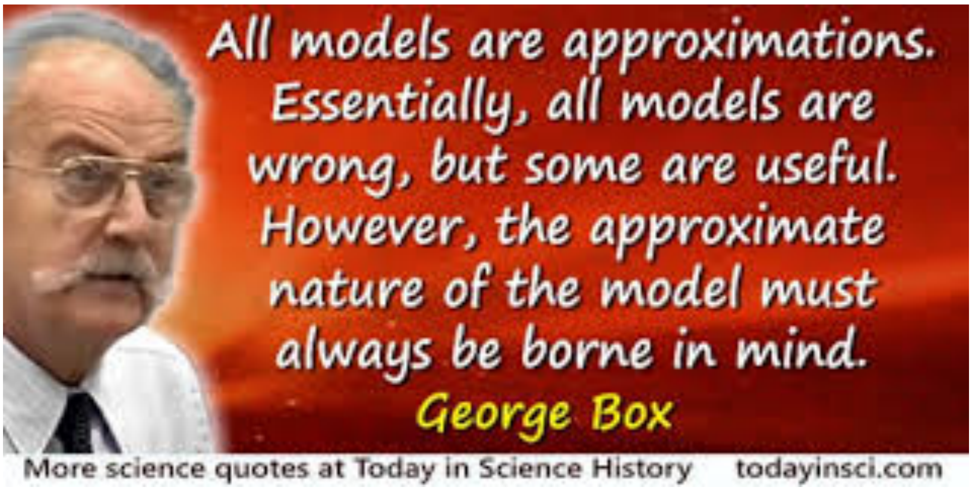
A Guess

(Q) Do we practically need any Partial Differential Equation?

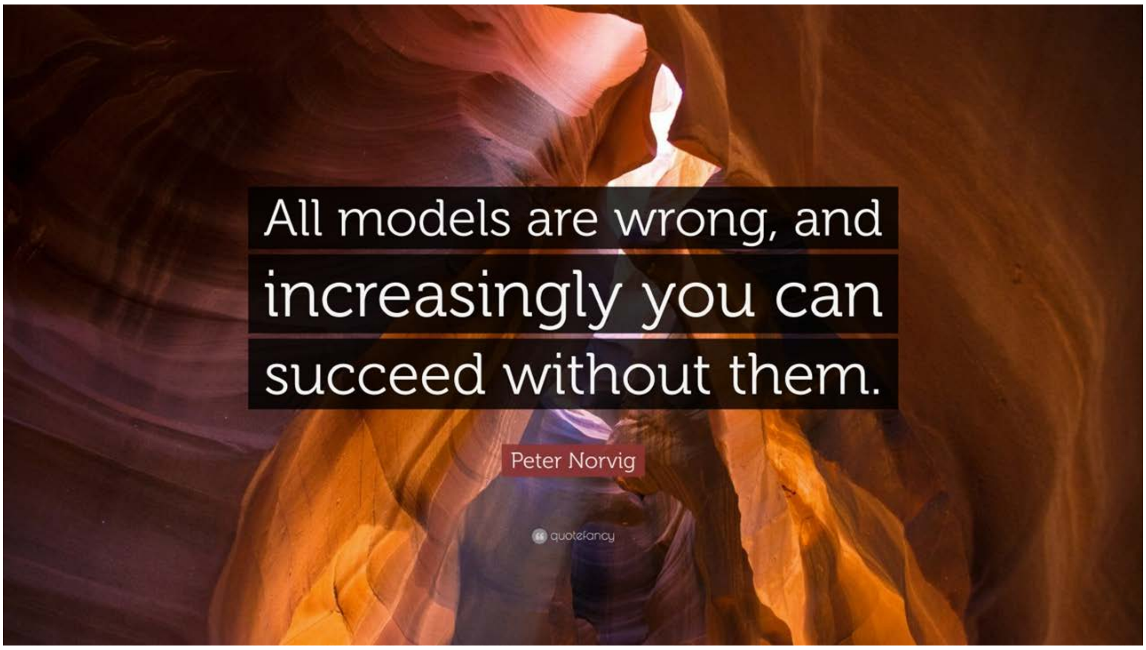
(A) No! Practically we need only the Boundary Conditions.

PDE would become an OLD Mistress

Should we rely on the Newtonian paradigm in the future?



Peter Norvig
The Director of Research at Google Inc



Hydra-zation

- Clairaut's Theorem, Young's theorem, Schwarz's theorem
- Diffusion/Heat Transfer Equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)$$

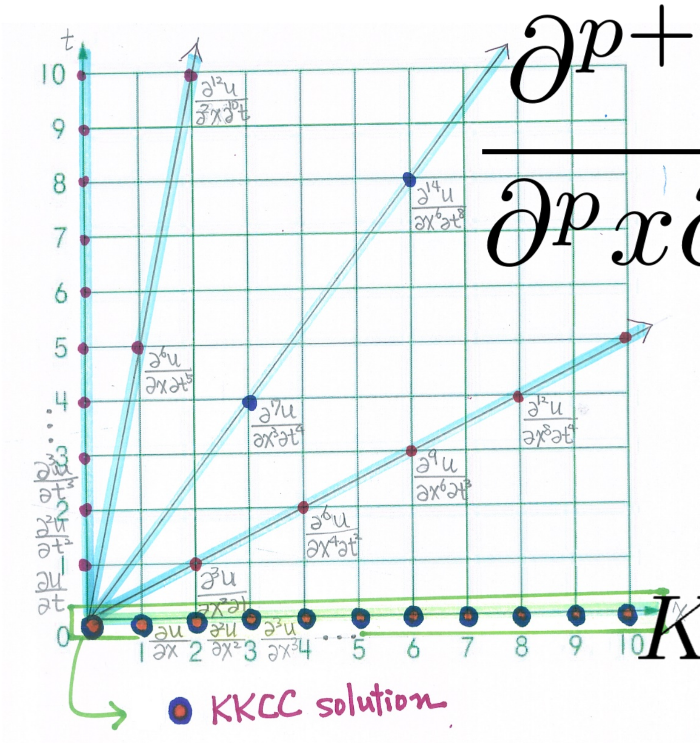
- If $u(x,t)$ is a solution to the PDE, so are

$$\frac{\partial^{p+q} u(x,t)}{\partial^p x \partial^q t}, \quad (p, q \in Z_{\geq 0}) .$$

- The solutions are independent.
- We can apply this result to any linear PDE.
- Also, we can apply a modified version to any nonlinear PDE.

Hydra-zation

Completeness on State



$$\frac{\partial^{p+q}u}{\partial^p x \partial^q t}$$

$$K^2C^2$$

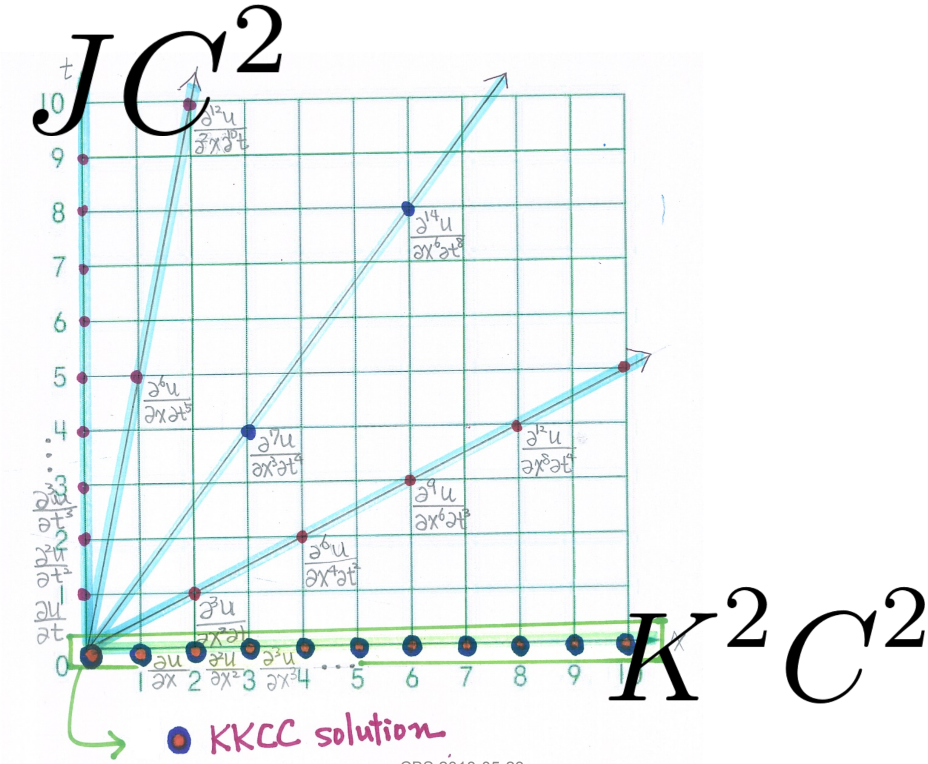
Occam's Razor

- Principle of Parsimony
- Non-Separable Solution

KISS

Keep. It. Simple. Stupid.

Completeness on State & Time



1. We need at least one solution before Hydra-zation!

2. To circumvent the Newtonian paradigm, we looking for a basis for solutions of all (linear) PDEs.

Now, we propose the following method for the two purposes above.

ODE and Eigen-equation

- Consider

$$y''(x) - (\lambda_1 + \lambda_2)\, y'(x) + \lambda_1 \lambda_2 y(x) = 0.$$

- Eigen-equation through the Fundamental Theorem of Algebra (FTA)

$$\lambda^2 - (\lambda_1 + \lambda_2)\,\lambda + \lambda_1 \lambda_2$$
$$= (\lambda - \lambda_1)\,(\lambda - \lambda_2) = 0$$

- Solution for $\lambda_1 \neq \lambda_2$

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

PDE and Eigen-equation 1

- Consider
$$u - \frac{1}{\lambda_1}u_x - \frac{1}{\theta_1}u_t + \frac{1}{\lambda_1\theta_1}u_{xt} = 0.$$
- Eigen-equation:
$$1 - \frac{1}{\lambda_1}\lambda - \frac{1}{\theta_1}\theta + \frac{1}{\lambda_1\theta_1}\lambda\theta = \left(1 - \frac{1}{\lambda_1}\lambda\right)\left(1 - \frac{1}{\theta_1}\theta\right) = 0.$$
- Solution: Linear combination of the PDE's
$$u = \frac{1}{\lambda_1}u_x, \quad u = \frac{1}{\theta_1}u_t$$
- Thus,
$$u(x,t) = C_1(t)e^{\lambda_1 x} + C_2(x)e^{\theta_1 t}.$$

PDE and Eigen-equation 2

- Consider
$$u - \frac{1}{\lambda_1}u_x - \frac{1}{\theta_1}u_t = 0.$$
- Eigen-equation:
$$1 - \frac{1}{\lambda_1}\lambda - \frac{1}{\theta_1}\theta = 0.$$
- (Q) Can we factor-out this eigen-equation?
  Does the FTA for 2D polynomials exist?
  (A) Yes!

A Fundamental Theorem of Algebra, Spectral Factorization, and Stability of Two-Dimensional Systems (CBS, 2003)

Abstract—In his doctoral dissertation in 1797, Gauss proved the fundamental theorem of algebra, which states that any one-dimensional (1-D) polynomial of degree n with complex coefficients can be factored into a product of n polynomials of degree 1. Since then, it has been an open problem to factorize a two-dimensional (2-D) polynomial into a product of basic polynomials. Particularly for the last three decades, this problem has become more important in a wide range of signal and image processing such as 2-D filter design and 2-D wavelet analysis. In this paper, a fundamental theorem of algebra for 2-D polynomials is presented. In applications such as 2-D signal and image processing, it is often necessary to find a 2-D spectral factor from a given 2-D autocorrelation function. In this paper, a 2-D spectral factorization method is presented through cepstral analysis. In addition, some algorithms are proposed to factorize a 2-D spectral factor finely. These are applied to deriving stability criteria of 2-D filters and nonseparable 2-D wavelets and to solving partial difference equations and partial differential equations.

PDE and Eigen-equation 3

- Consider a linear PDE

$$\sum_{a=0}^{M}\sum_{b=0}^{N}\alpha_{a,b}\frac{\partial^{a+b}u(x,t)}{\partial^a x\,\partial^b t}=0,\quad \alpha_{0,0}=1.$$

- Eigen-equation:

$$\sum_{a=0}^{M}\sum_{b=0}^{N}\alpha_{a,b}\lambda^a\theta^b=0.$$

- The 2D FTA implies

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\alpha_{i,j}\lambda^i\theta^j=\prod_{a=0}^{\infty}\prod_{b=0}^{\infty}\left(1-\frac{1}{\nu_{a,b}}\lambda^a\theta^b\right)$$

PDE and Eigen-equation 4

- Let $u_{a,b}(x,t)$ be a solution of PDE for $(a,b)-th$ factor

$$\frac{\partial^{a+b} u_{a,b}(x,t)}{\partial^a x\, \partial^b t} = \nu_{a,b} u_{a,b}(x,t).$$

- Then,

$$u(x,t) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} C(a,b) u_{a,b}(x,t).$$

- For example,

$$u_{a,b}(x,t) = \exp\left(\nu_1 x + \nu_2 t\right)$$

with $\nu_{a,b} = \nu_1^a \nu_2^b.$

To Apply this Method to Diffusion Equation

- Consider a diffusion equation $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$.
- Let $v(x,t) = e^{\beta t} u(x,t).$
- Then, $\frac{v_t}{v} = \frac{u_t}{u} + \beta, \quad \frac{v_x}{v} = \frac{u_x}{u}, \quad \frac{v_{xx}}{v} = \frac{u_{xx}}{u}.$

- Thus, for any $\beta(\neq 0)$

$$v - \frac{1}{\beta} v_t - \frac{1}{2\beta} v_{xx} = 0.$$

- The eigen-equation is

$$1 - \frac{1}{\beta}\lambda - \frac{1}{2\beta}\theta^2 = 0.$$

Solution by Integration (?)

- The current solution w/ initial condition $u(x,0)$ is

$$u(x,t) = \int_{-\infty}^{\infty} u(x-w,0)\,\frac{1}{\sqrt{t}}\phi\left(\frac{w}{\sqrt{t}}\right)dw.$$

- (Some examples)

$$(1)\quad \int_{-\infty}^{\infty} \delta(x-w)\,\frac{1}{\sqrt{t}}\phi\left(\frac{w}{\sqrt{t}}\right)dw = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)$$

$$(2)\quad \int_{-\infty}^{\infty} 1_{(-\infty,x]}(w)\,\frac{1}{\sqrt{t}}\phi\left(\frac{w}{\sqrt{t}}\right)dw = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

$$(3)\quad \int_{-\infty}^{\infty} (x-w)1_{(-\infty,x]}(w)\,\frac{1}{\sqrt{t}}\phi\left(\frac{w}{\sqrt{t}}\right)dw$$

$$= x\Phi\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\tau}\phi\left(\frac{x}{\sqrt{\tau}}\right)$$

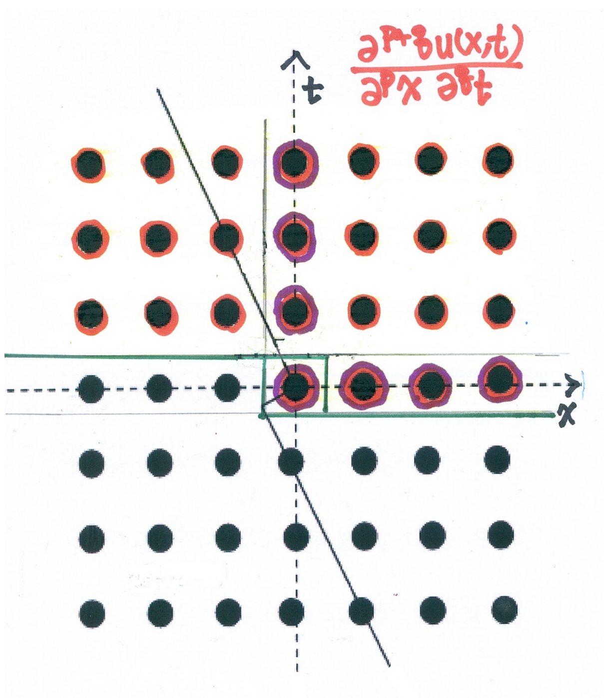
Integrated Solution

- When $u(x,t)$ is a solution of the diffusion PDE, is $\int_{-\infty}^{x} u(z,t)dz$ another one?

- Possible when $u(-\infty,t) \equiv 0.$

- When we use an integrated solution, we would rather make the system anticipative. Otherwise, an identification problem arises. To do it, we may use a Non-Symmetric Half Plane (NSHP).

Non-Symmetric Half Plane



Including Jumps (Singularities)

- Let $a_n \equiv \int_{-\infty}^{\infty} f\left(\frac{x}{\sqrt{t}}\right) \frac{1}{t^{(-n+1)/2}} He_n\left(\frac{x}{\sqrt{t}}\right) dx.$
- Then $\sum_{n=0}^{N} a_n \frac{1}{t^{(n+1)/2}} He_n\left(\frac{x}{\sqrt{t}}\right) \phi\left(\frac{x}{\sqrt{t}}\right)$

  converges to $f\left(\frac{x}{\sqrt{t}}\right)$ in L^2- sense.
- If f is smooth or piecewise smooth, then the series converges pointwisely.
- In latter case, the series the series converges to $$\frac{f(z+)+f(z-)}{2},$$ when z is a point of discontinuity.

Including Jumps (Singularities)

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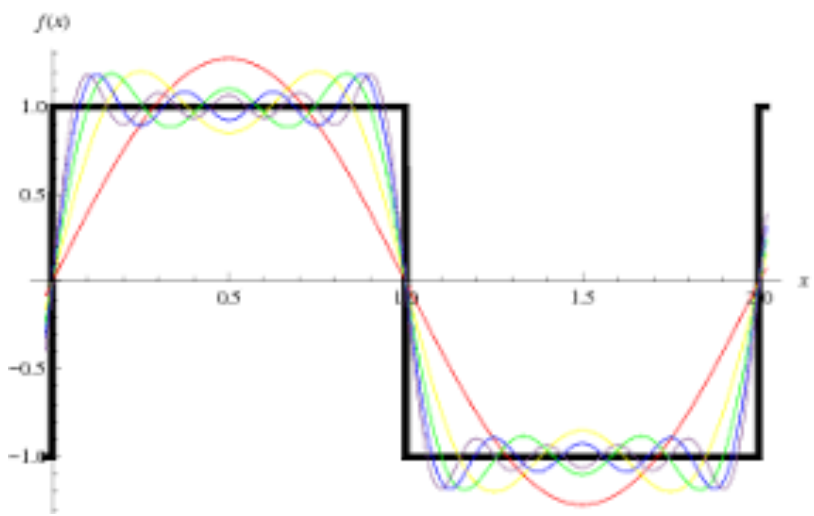
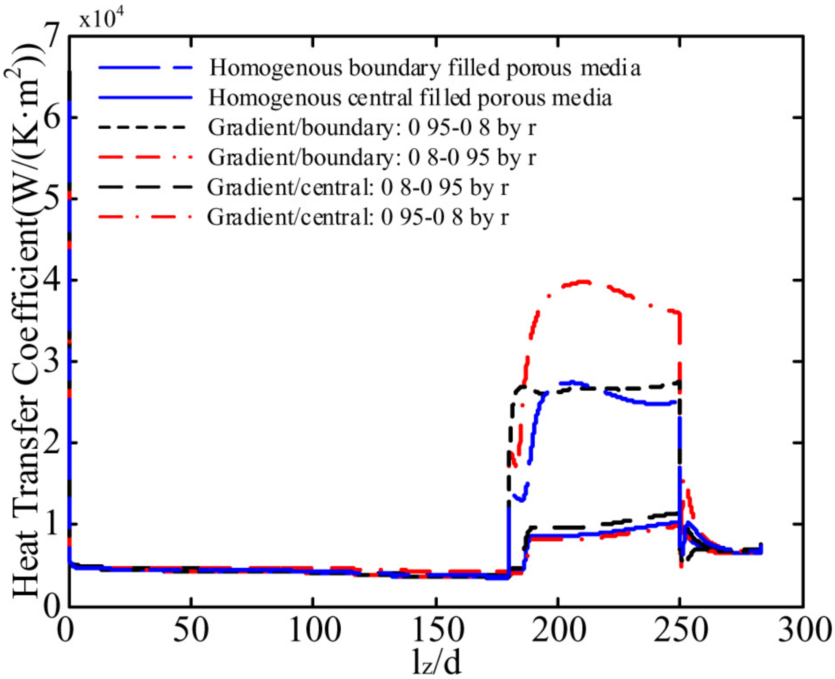
  when z is a point of discontinuity.

(Q) Gibbs Phenomenon

Gibbs Phenomenon

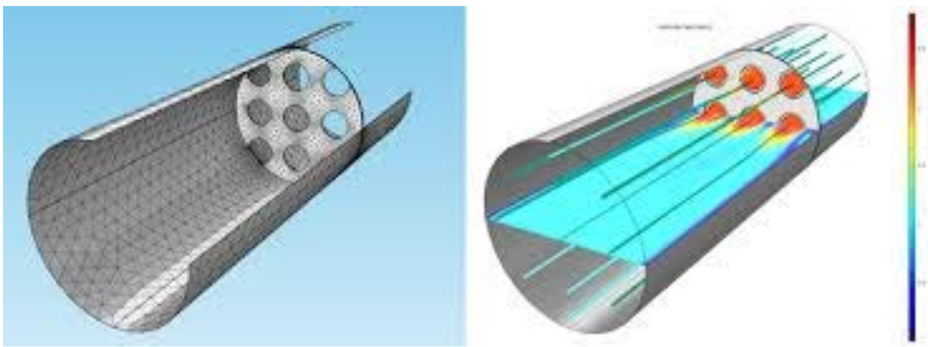


Heat transfer analysis of transcritical hydrocarbon fuel flow in a tube partially filled with porous media



Partially filled with porous media

- l_z the length of headed section (meters)
- d tube diameter (meters)



Binary (aka Digital) Option

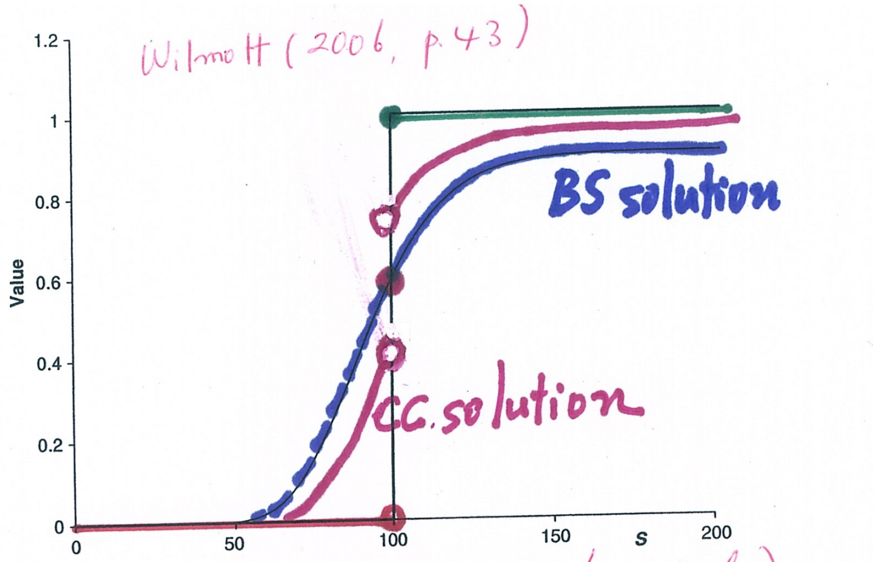
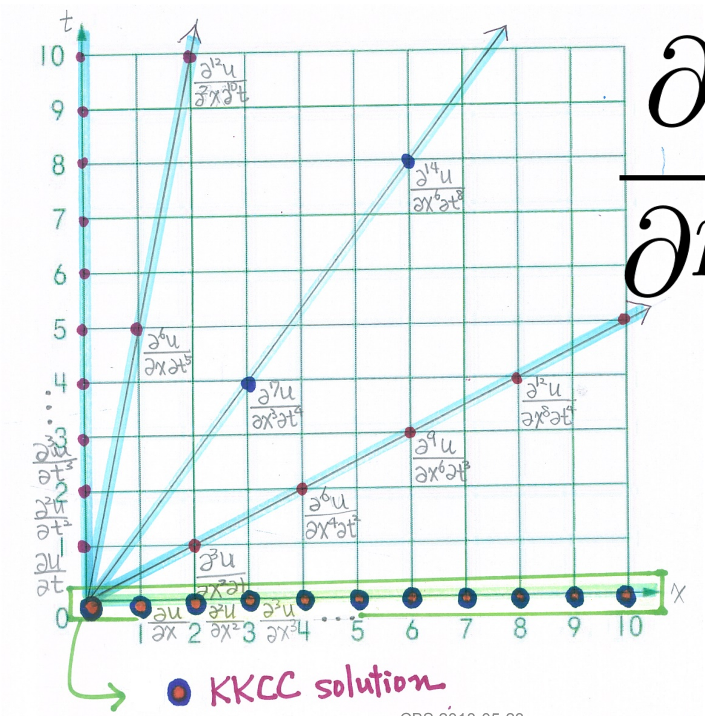


Figure 2.15 Payoff diagram for a binary call option. (Binary / Digital)

Completeness & Jumps



$$\frac{\partial^{p+q} u}{\partial^p x \partial^q t}$$

Feynman-Kac Formula

- PDE

$$-\frac{\partial v}{\partial t} + kv = \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 v}{\partial x_i^2} + g; \quad \text{on } R^d \times [0, T]$$

$$u(x, T) = f(x); \quad \left(x \in R^d\right)$$

- Expectation

$$u(x,t) = E^x\left(f\left(W_{T-t}\right)\exp\left(-\int_0^{T-t} k(W_s)ds\right)\right) + E^x\left(\int_0^{T-t} g\left(t+\theta, W_\theta\right)\exp\left(-\int_0^{\theta} k(W_s)d\theta\right)\right)$$

New Formula by Choi & Choi (2018)

- Let $d=1, \quad k(z)\equiv 0, \quad g(z)\equiv 0.$
- Let
$$h_M^{\boldsymbol{\zeta}}(w,t)\equiv \sum_{m=0}^{M}\zeta_m \frac{1}{t^{m/2}}He_m\left(\frac{w}{\sqrt{t}}\right).$$
- Then, a generalized solution is
$$u^{\boldsymbol{\zeta}}(x,t)=\int_{-\infty}^{\infty}u(x-w,0)\,h_M^{\boldsymbol{\zeta}}(t,w)\frac{1}{\sqrt{t}}\phi\left(\frac{w}{\sqrt{t}}\right)dw.$$
- This equals to
$$\begin{aligned}u^{\boldsymbol{\zeta}}(x,t)&=E^0\Big(u\left(x-W_t,0\right)\big)\,h_M^{\boldsymbol{\zeta}}(t,W_t)\Big)\\&=\zeta_0 u(x,t)+\sum_{m=1}^{M}\zeta_m\frac{1}{t^{m/2}}E^0\left(u\left(x-W_t,0\right)He_m\left(\frac{W_t}{\sqrt{t}}\right)\right).\end{aligned}$$

Coefficients & Initial Condition

- The coefficients $\zeta_0,\zeta_1,\zeta_2,\cdots$ should satisfy
$$E^0\left(\sum_{m=0}^{M}\zeta_m\frac{1}{t^{m/2}}u\left(x-W_t,0\right)He_m\left(\frac{W_t}{\sqrt{t}}\right)\right)=u(x,0).$$

An Example

- $$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}; \quad \text{on } R \times [0,T]$$
  $$u(x,0) = \exp\left(-|x|\right); \quad x \in R$$

- For any $\eta_1 \in R$,

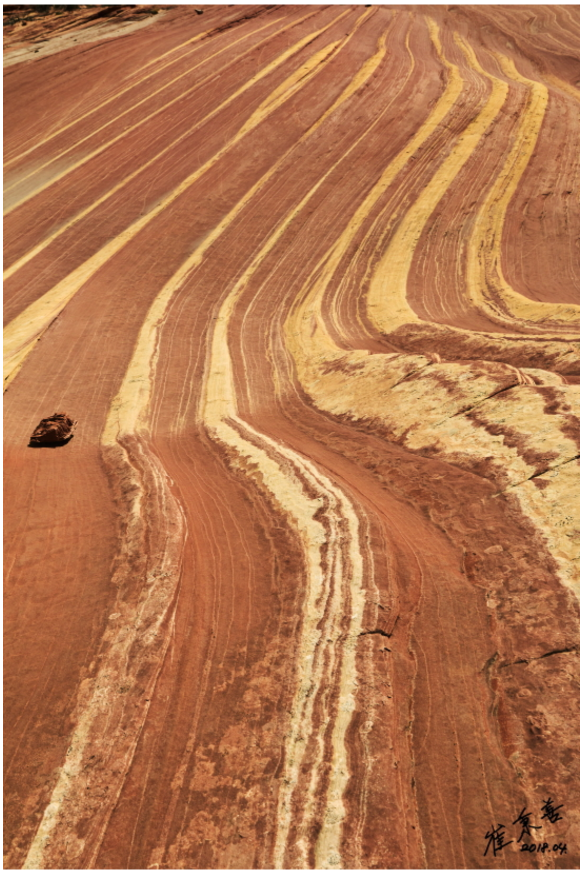
$$u^{\zeta}(x,t) = u(x,t) + \eta_1 \left\{ \exp\left(-x + \frac{t}{2}\right) \Phi\left(\frac{x-t}{\sqrt{t}}\right) 1_{(-\infty,0)}(x) + \exp\left(x + \frac{t}{2}\right) \Phi\left(-\frac{x+t}{\sqrt{t}}\right) 1_{(0,\infty)}(x) \right\}$$

where

$$u(x,t) = \exp\left(x + \frac{t}{2}\right) \Phi\left(-\frac{x+t}{\sqrt{t}}\right) + \exp\left(-x + \frac{t}{2}\right) \Phi\left(\frac{x-t}{\sqrt{t}}\right).$$

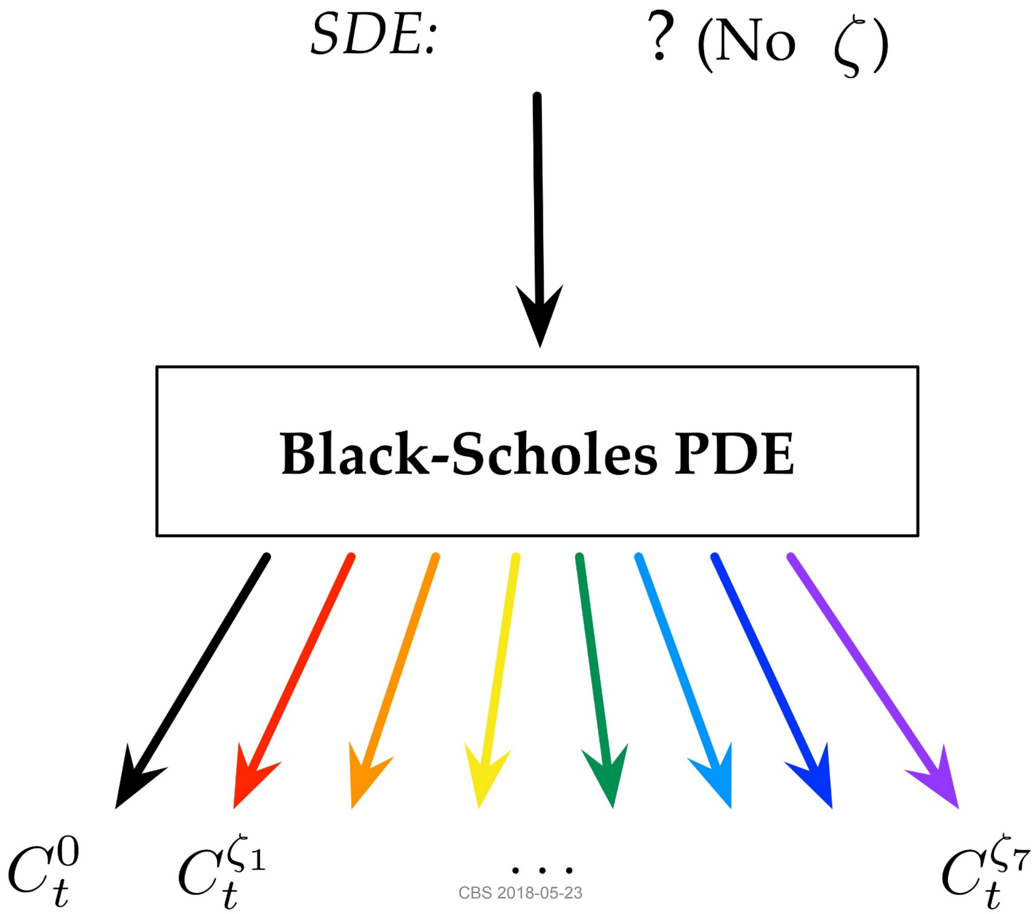
Very Serious Problems

- Monte Carlo Simulations of Diffusion/Heat Transfer Problem et al.
- Risk Neutral Option Pricing
- If observations do not come from an exact Gauss distribution, the expectation is not correct and can have infinitely many values. Moreover, the Gaussian assumption is impractical.

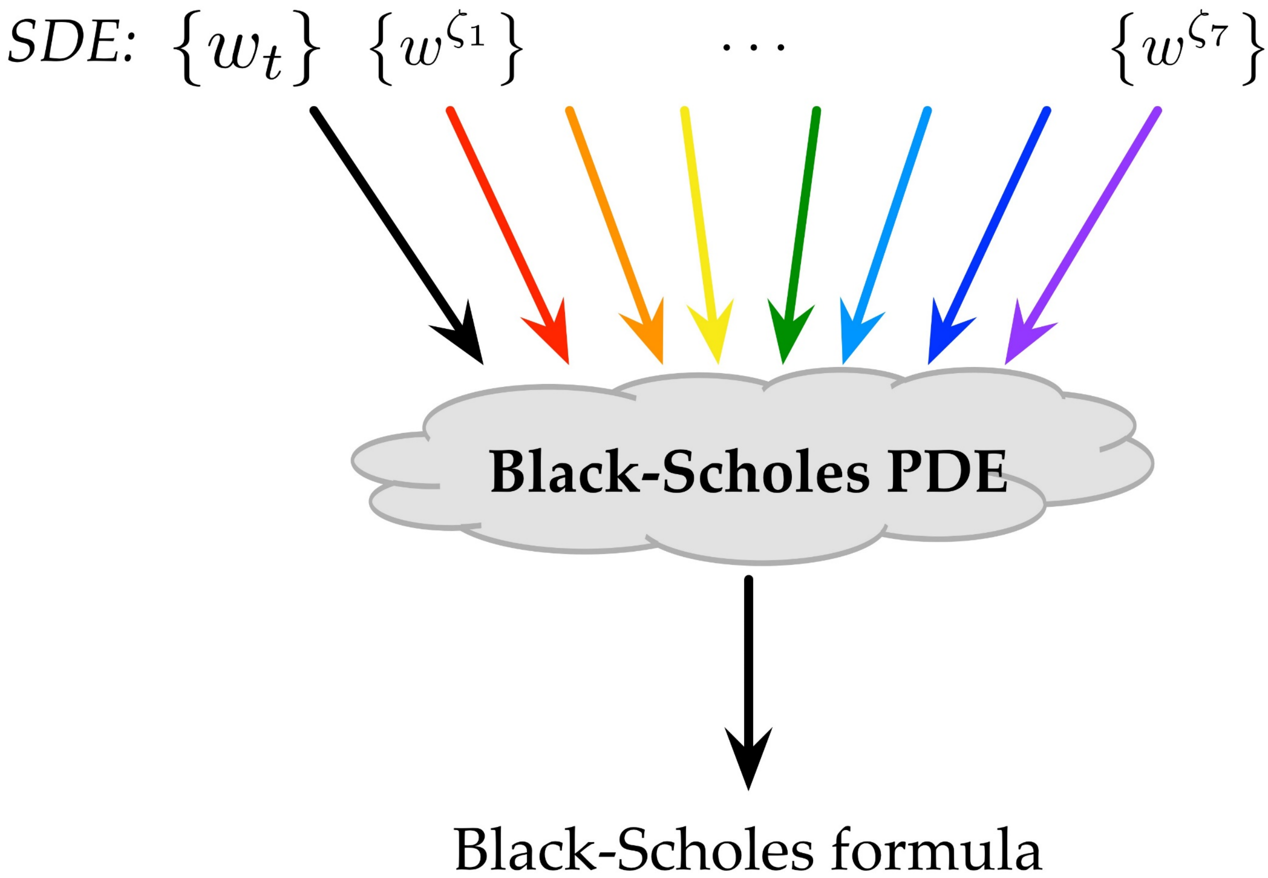


How can we challenge this problem in the future?

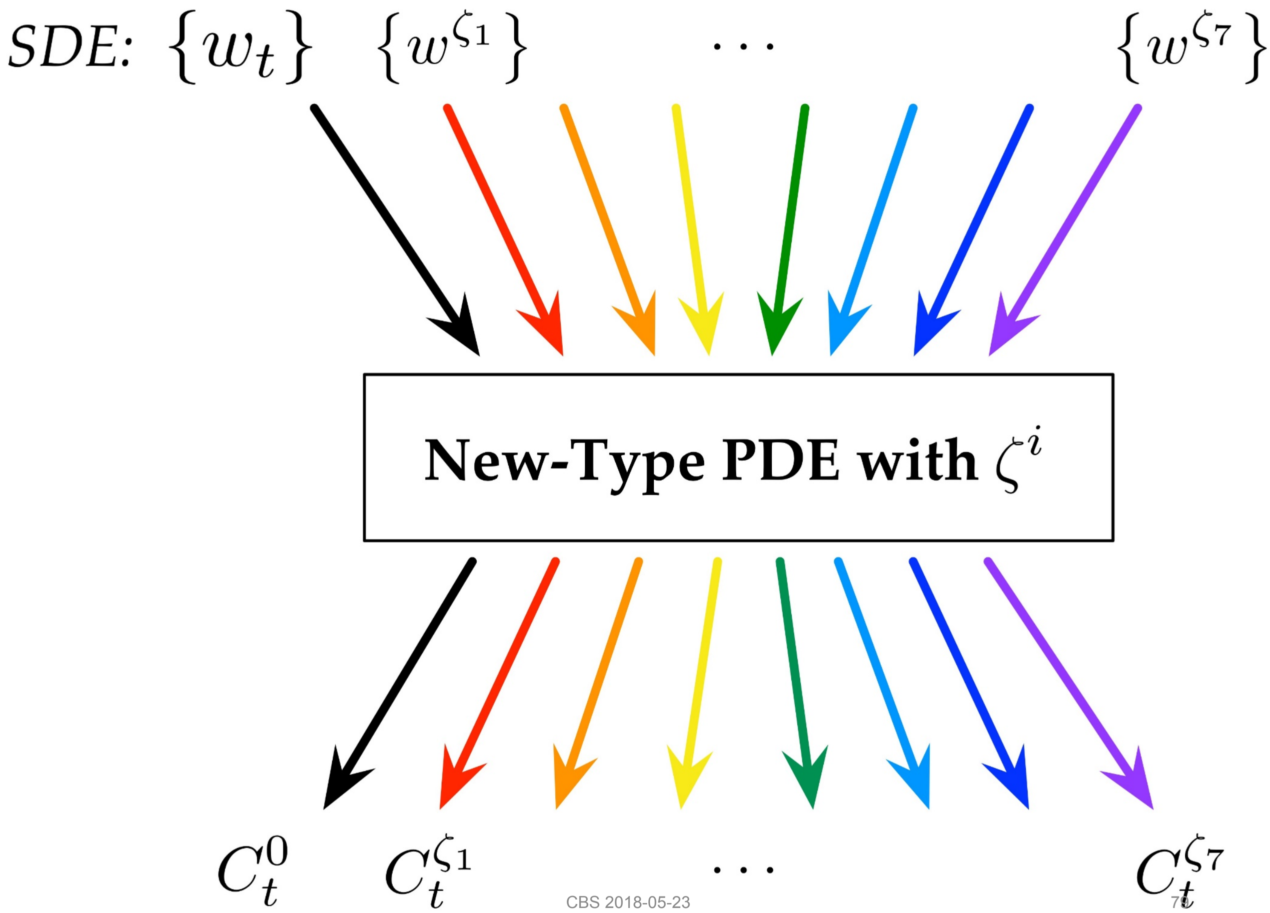
- Establish a new stochastic calculus instead of either Ito calculus or Lévy driven stochastic calculus.
- Wish for Kim's Calculus, Park's Calculus, Lee's Calculus, …



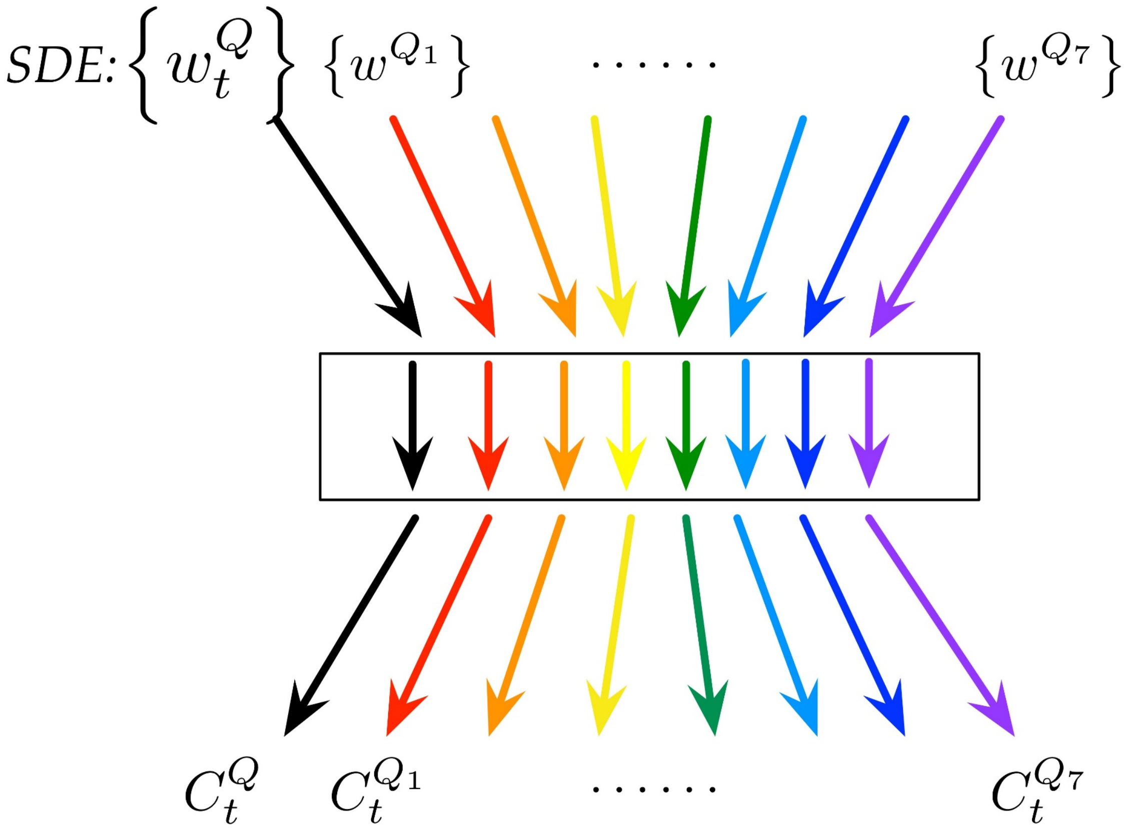
SDE: ? (No ζ)
Black-Scholes PDE
C_t^0 $C_t^{\zeta_1}$ $\cdots$ $C_t^{\zeta_7}$



SDE: $\{w_t\}$ $\{w^{\zeta_1}\}$ $\cdots$ $\{w^{\zeta_7}\}$
Black-Scholes PDE
Black-Scholes formula



SDE: $\{w_t\}$ $\{w^{\zeta_1}\}$ $\cdots$ $\{w^{\zeta_7}\}$
New-Type PDE with ζ^i
C_t^0 $C_t^{\zeta_1}$ $\cdots$ $C_t^{\zeta_7}$



SDE: $\left\{w_t^Q\right\}$ $\{w^{Q_1}\}$ $\cdots\cdots$ $\{w^{Q_7}\}$
C_t^Q $C_t^{Q_1}$ $\cdots\cdots$ $C_t^{Q_7}$

Infinite Divisibility

- A CDF F is infinitely divisible (InfDiv) if any n, there exist IID RV $X_{n1}, X_{n2}, \cdots, X_{nn}$ such that

$$X_{n1} + X_{n2} + \cdots + X_{nn} \overset{d}{\sim} F\,.$$

- Bruno de Finetti (1929)

Lévy Process

- InfDiv CDF corresponds in a natural way to a Lévy Process.
- A Lévy process is a stochastic process with stationary independent increments.
- Let $\{l_t\}$ be a Lévy process. Then, RV l_t is InfDiv.
- If F is InfDiv, a Lévy process $\{l_t\}$ is constructed from it. For any interval $[s, t]$ where $t - s = p/q$, we can define $l_t - l_s$ to have the same CDF as $X_{q,1} + X_{q,2} + \cdots + X_{q,p}$. When $t - s$ is irrational, we use a continuity argument.

Stable Distribution

- A CDF is stable if a linear combination of two RVs with the CDF has the same CDF up to location and scale parameters.
- An RV is stable if its CDF is stable.
- (aka) the Lévy alpha-stable distribution

Heavy-Tail Distribution
$\alpha -$ Stable Distribution

$$\varphi(t) = \begin{cases} \exp\left[-\sigma|t|\left(1 + i\beta\frac{2}{\pi}\operatorname{sign}(t)\log|t|\right) + i\mu t\right], \\ \qquad \alpha = 1, \\ \exp\left[-\sigma^{\alpha}|t|^{\alpha}\left(1 - i\beta\operatorname{sign}(t)\tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu t\right], \\ \qquad \alpha \neq 1. \end{cases} \tag{7}$$

Stable Distribution

- $0 < \alpha \leq 2$
- $\alpha = 2$: Gaussian distribution
- $\alpha = 1$: Cauchy distribution
- $\alpha = 0.5$: Lévy distribution

$$\phi(\theta) = \exp\left(-\sqrt{-2ic\theta} + i\mu\theta\right)$$

$$F(x) = \sqrt{\frac{c}{2\pi}}(x-\mu)^{-3/2}\exp\left(-\frac{c}{2(x-mu)}\right)$$

- $\alpha < 2 \Rightarrow$ No variance
- $\alpha \leq 1 \Rightarrow$ No mean
- Guess that a Gram-Charlier distribution does not belong to the family of stable distributions, but is asymptotically similar to a stable distribution with α near 2.

Stable Paretian Distributions

- CLT: normed sum of a set of RVs, each with finite variance, tends towards a Gaussian CDF as number of RVs increases.
- Without the finite variance assumption, the limit may be a stable distribution that is not Gaussian.
- Mandelbrot (1961, Econometrica) called them "stable Paretian distributions“.
- Those maximally skewed in the positive direction with 1 < α < 2 are called "Pareto–Lévy distributions", which Mandelbrot regarded as better descriptions of stock and commodity prices than normal distributions.[

Cootner (1964, p. 337)

should be considered explicitly. Mandelbrot, like Prime Minister Churchill before him, promises us not utopia but blood, sweat, toil, and tears. If he is right, almost all of our statistical tools are obsolete — least squares, spectral analysis, workable maximum-likelihood solutions, all our established sample theory, closed distribution functions. Almost without exception, past econometric work is meaningless. Surely, before consigning centuries of work to the ash pile, we should like to have some assurance that all our work is truly useless. If we have permitted ourselves to be fooled for as long as this into believing that the Gaussian assumption is a workable one, is it not possible that the Paretian revolution is similarly illusory? At any rate, it would seem desirable not only to

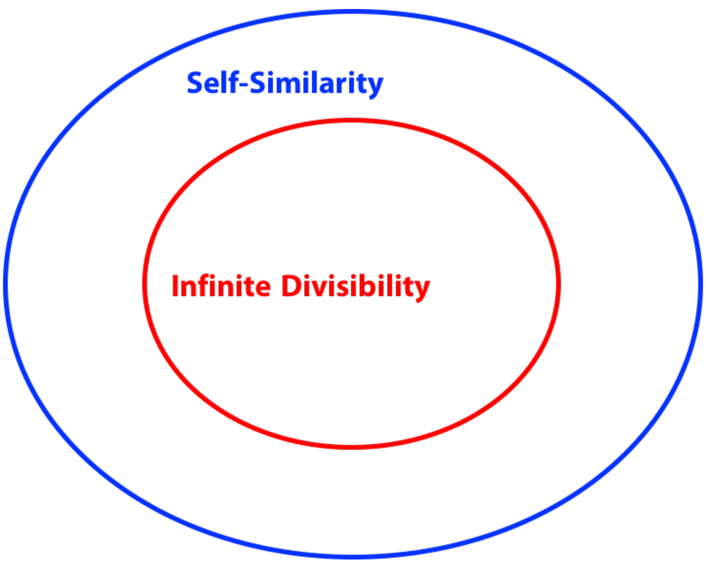
Infinite Divisibility and Stable

- Stable CDF $\Rightarrow$ InfDiv CDF
- InfDiv CDF doss NOT imply Stable CDF.
- Counterexample: Poisson distribution.

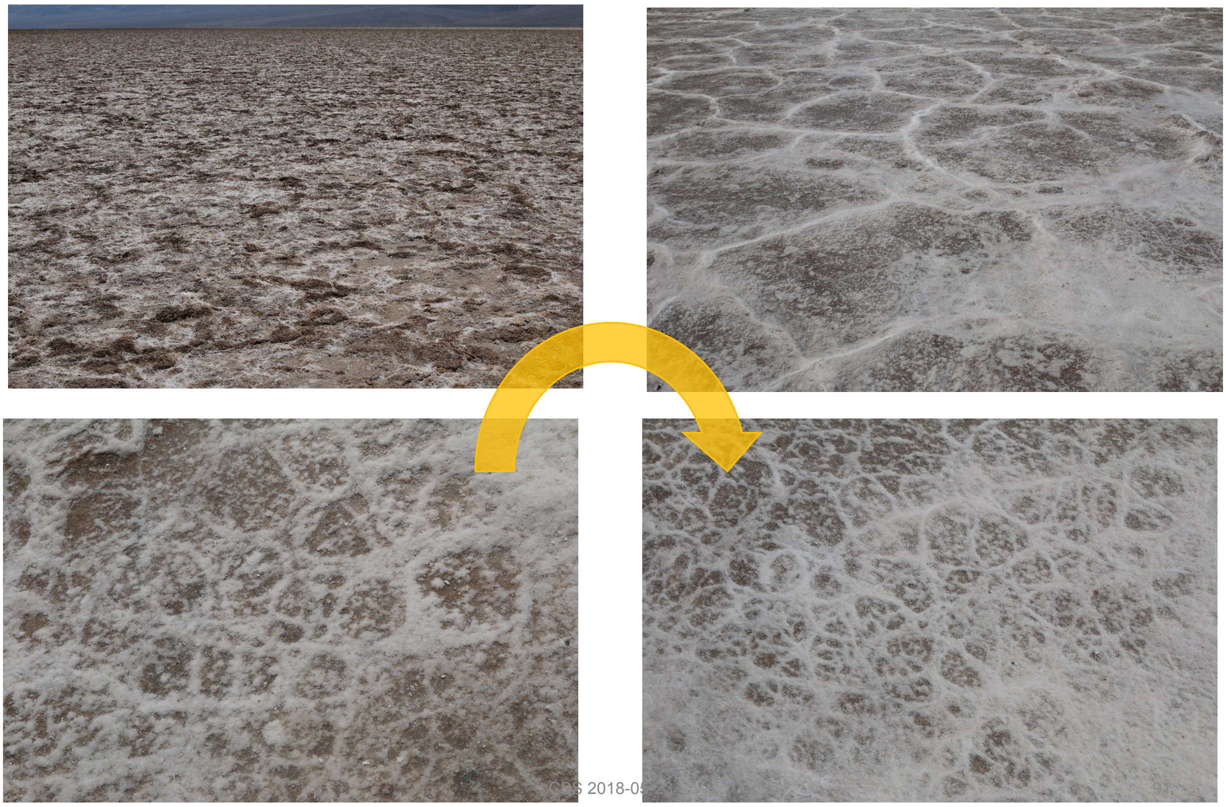
  For each λ>0 and each n, let $X_1, X_2, \cdots, X_n \overset{d}{\sim} Poisson(\lambda/n)$.

  Then $\sum_{i=1}^{n} X_i \overset{d}{\sim} Poisson(\lambda)$.

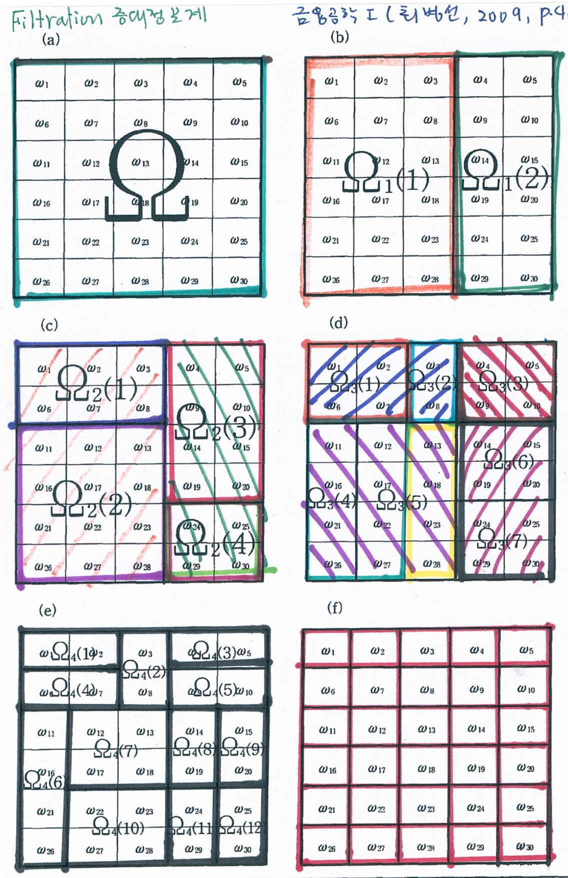
  However, $X_1 + \cdots + X_{n-1} + \pi X_n$ does not have the Poisson distribution.



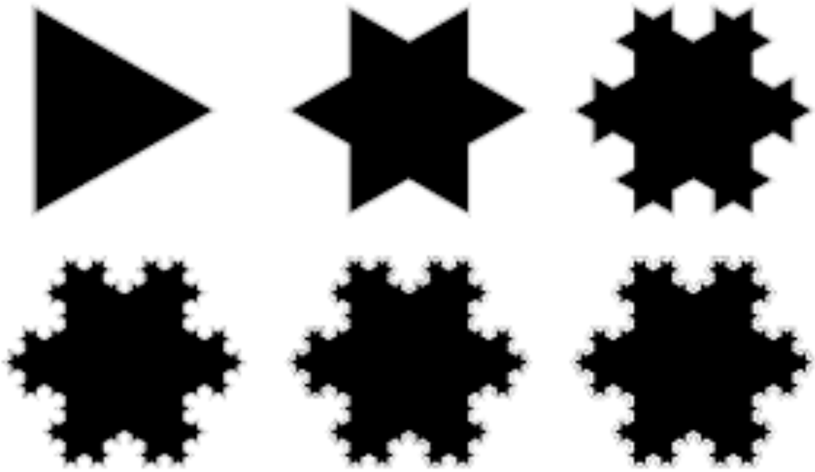
Self-Similarity@Bad Water Basin

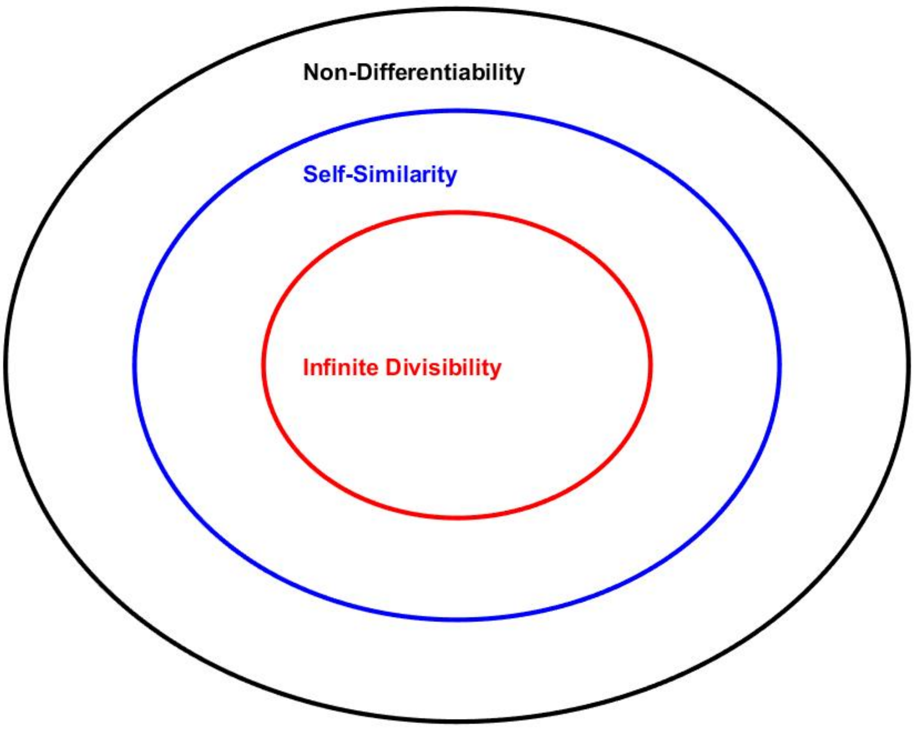


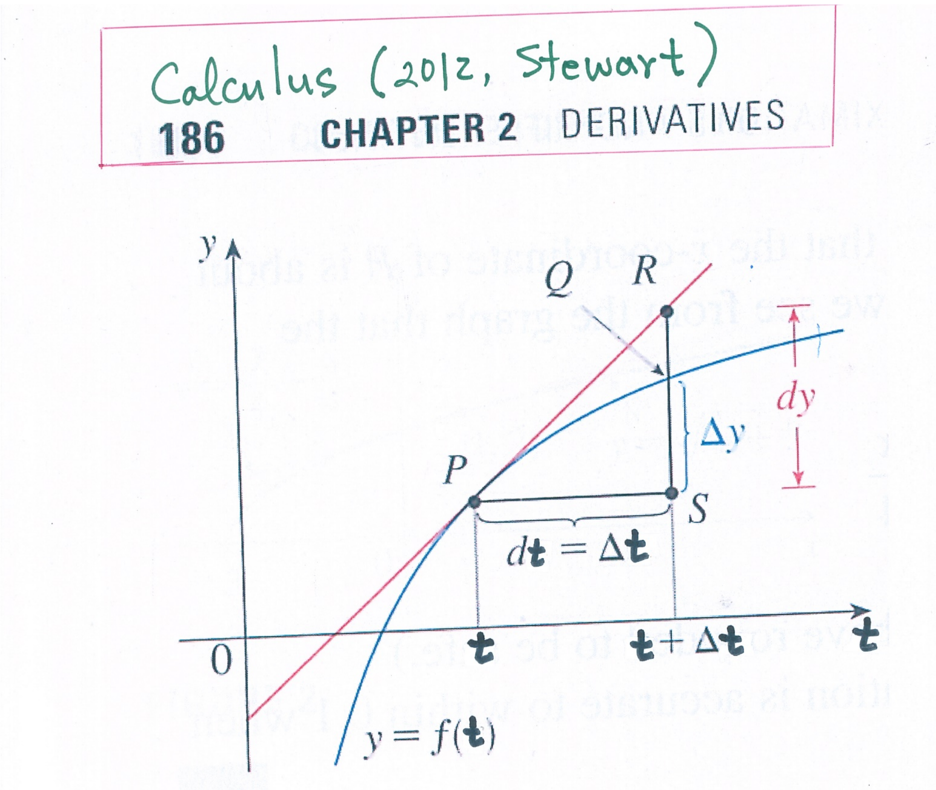
Filtration 증대정보계   금융공학 I (최병선, 2009, p.463)



Koch Snowflake (1904)






Calculus (2012, Stewart)
186 CHAPTER 2 DERIVATIVES
y
Q
R
dy
Δy
P
S
dt = Δt
0
t
t + Δt
t
y = f(t)


崔秉善
2018.04

The Second Law of Our Lives

- If all else fails,

- However, there is

no  key

in our real lives. (CBS)

To be continued …

Infinitely Many Solutions to the Black-Scholes PDE; Information Theory Point of View

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