Nobel Prize in Physics 2016

물질의 위상수학적 성질
Topological Properties of Matter

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Nobel Prize in Physics 2016

½ David J. Thouless

½ J. Michael Kosterlitz and F. Duncan M. Haldane

"*theoretical discoveries of topological phase transitions and topological phases of matter*"

David James Thouless (1934 -) *University of Washington*

Born in Scotland, Studied nuclear physics under the supervision of H. Bethe

Won Maxwell prize, Wolf prize, Onsager prize, Nobel prize

Phase transition: Thouless effect, Berezinskii-Kosterlitz-Thouless (BKT) transition

Localization: Thouless picture (energy)

Spin-glass: de Almeida-Thouless (AT) line, Thouless-Anderson-Palmer (TAP) approach

Quantum Hall effect: topological quantization

Spectrum of the Schrödinger operator: wave function scaling

Vortex dynamics: quantum vortex

- Prologue
- Symmetry and Phase Transition
- Berezinskii-Kosterlitz-Thouless Transition
- Quantum Hall Effect
- Vortex Dynamics
- Epilogue

Prologue

Elementary particle (← *Field theory*)

symmetry, unification, charm, beauty, TOE,…

Condensed matter (← *Statistical mechanics*)

symmetry-breaking, disorder, randomness, frustration, chaos,…

Yet best precision comes from "dirty" condensed matter!

voltage standard: Josephson effect $V_{dc} \equiv \langle V \rangle = \dfrac{\hbar}{2e}\dot{\phi} = n\dfrac{\hbar\omega}{2e}$ $(\square\ 10^{-17})$

resistance standard: quantum Hall effect $\sigma_H = n\dfrac{e^2}{h}$ $(\square\ 10^{-10})$

Why?

Quantum Numbers: Symmetry vs Topology

Symmetry → conserved quantity Noether's theorem

discrete eigenvalues for the operator quantum numbers

symmetry: fragile, subject to perturbation

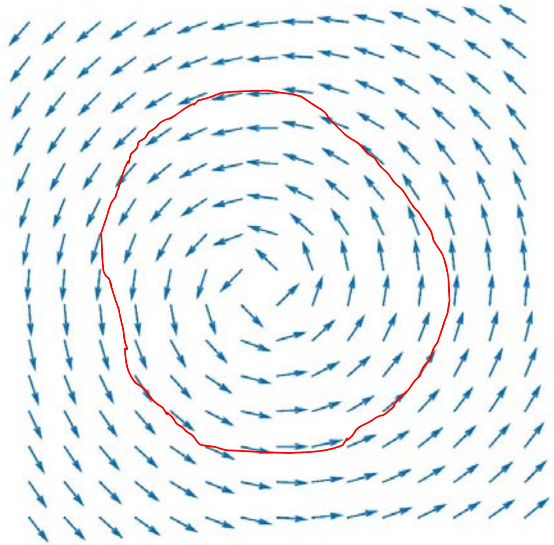
broken → mixing of the quantum numbers

Topology → winding numbers

topological charge (quantum numbers)

topology: robust against perturbation

→ high precision



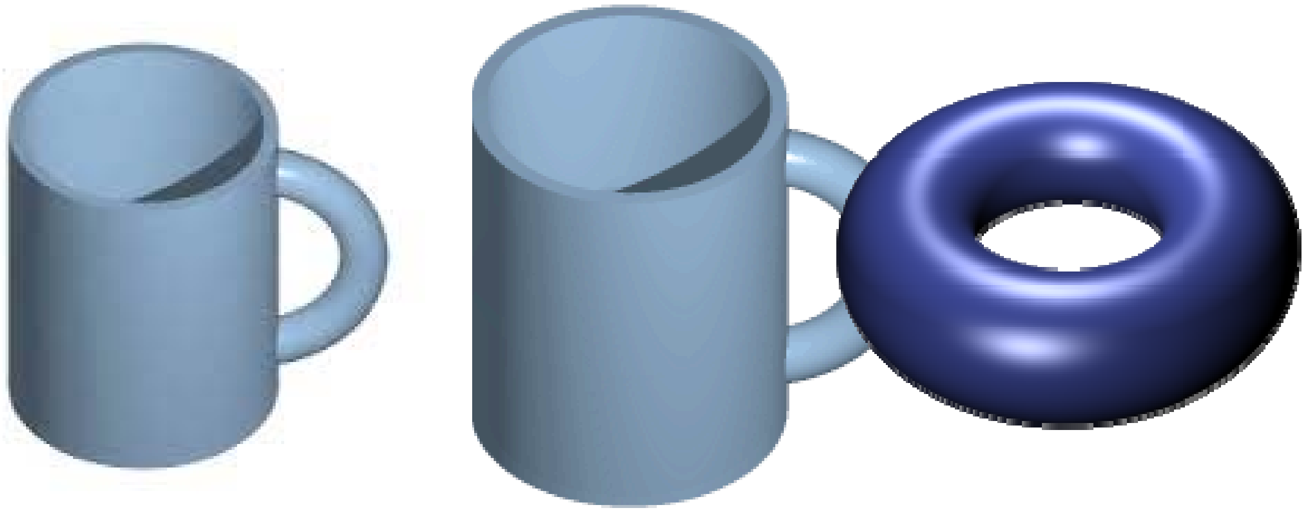
Condensate wave function

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\phi(\mathbf{r})}$$

$$\oint \nabla\phi \cdot d\mathbf{l} = 2\pi n$$ topological charge $n > 0$: vortex

Symmetry vs Topology

Mug and doughnut: different symmetry but the same topology



Quantum Interference

- Moving charge in the presence of vector potential $\mathbf{A}$
  - Hamiltonian

    $$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2$$

    AB phase acquired

    $$\phi_{AB} = \frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{e}{\hbar c}\int \mathbf{B}\cdot d\mathbf{a} = \frac{e}{\hbar c}\Phi_{AB}$$

    gauge transformation: $\mathbf{A} \to \mathbf{A} + \nabla\Lambda \quad \Rightarrow \quad \psi \to \psi e^{i(e/\hbar c)\Lambda}$

  ⇨ Bohm-Aharonov effect

- Moving magnetic moment in the presence of scalar potential A_0
  (moving solenoid in the presence of charge)
  - Hamiltonian

    $$H = \frac{1}{2m}\left(\mathbf{p} - \frac{1}{c}\mathbf{E}\times\boldsymbol{\mu}\right)^2$$

    AC phase acquired

    $$\phi_{AC} = \frac{e}{\hbar c}\oint \mathbf{A}_{AC}\cdot d\mathbf{l} \quad \left(\mathbf{A}_{AC} \equiv \frac{1}{e}\boldsymbol{\mu}\times\mathbf{E}\right)$$

  ⇨ Aharonov-Casher effect

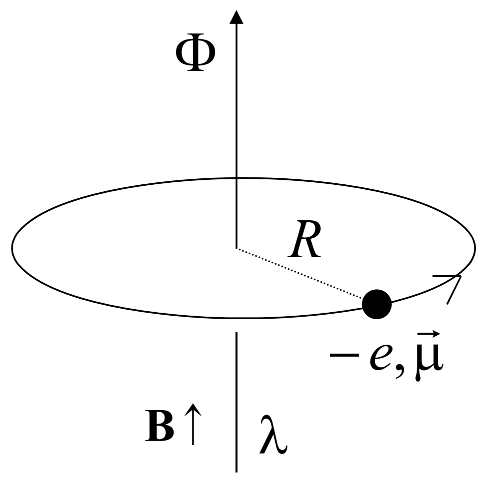
Persistent Currents

Free electrons in a metallic loop

- BA flux $\Phi_{AB} = \oint \mathbf{A}\cdot d\mathbf{l} = \int \mathbf{B}\cdot d\mathbf{a} \equiv f_{AB}\Phi_0 \qquad (\Phi_0 \equiv hc/e)$
- AC flux $\Phi_{AC} = \oint \mathbf{A}_{AC}\cdot d\mathbf{l} = \frac{\mu\sigma}{e}\int \nabla\cdot\mathbf{E}\, da \equiv \frac{\mu\sigma}{e} 4\pi\lambda \equiv \sigma f_{AC}\Phi_0$

  (σ = ±1: spin state; λ: linear charge density)
- Energy level $E_{n\sigma} = \frac{\hbar^2}{2mR^2}(n+f)^2 \quad (-1/2 < f \le 1/2) \quad f \equiv f_{AB} + \sigma f_{AC}$
- Total energy $E = \sum_{n\sigma} E_{n\sigma}$
- Charge current Spin current

$$I_c = -\frac{e}{2\pi\hbar}\frac{\partial E}{\partial f_{AB}} \qquad I_s = \frac{1}{4\pi}\frac{\partial E}{\partial f_{AC}}$$



Superfluids and Superconductors

Condensate wave function $\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\phi(\mathbf{r})}$

Superfluid velocity $\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi$

circulation $\oint \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m}\oint \nabla\phi \cdot d\mathbf{l} = n\frac{h}{m}$ winding number n: topological q. number

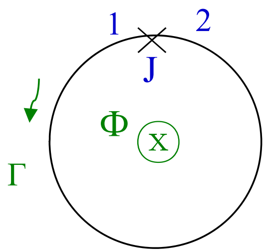
vortex quantization

Superconductor

$$\mathbf{j}_s = 2e\psi^* \mathbf{v}\psi = \frac{2e|\psi|}{m}\left(\hbar\nabla\phi - \frac{2e}{c}\mathbf{A}\right) = 0, \text{ inside superconductor} \quad \Rightarrow \quad \nabla\phi = \frac{2e}{\hbar c}\mathbf{A} = \frac{2\pi}{\Phi_0}\mathbf{A}$$

flux $\Phi = \oint \mathbf{A}\cdot d\mathbf{l} = \frac{\hbar c}{2e}\oint \nabla\phi \cdot d\mathbf{l} = 2n\pi\frac{\hbar c}{2e} \equiv n\Phi_0$ winding number n:

flux quantization



SQUID

$$\int_{1,\Gamma}^{2} \nabla\phi\cdot d\mathbf{l} = \oint \nabla\phi \cdot d\mathbf{l} - \int_{2,J}^{1} \nabla\phi\cdot d\mathbf{l} = 2n\pi - \phi \quad (\phi \equiv \phi_1 - \phi_2)$$

$$\int_{1,\Gamma}^{2} \mathbf{A}\cdot d\mathbf{l} = \oint \mathbf{A}\cdot d\mathbf{l} - \int_{2,J}^{1} \mathbf{A}\cdot d\mathbf{l} = \Phi - \int_{2,J}^{1} \mathbf{A}\cdot d\mathbf{l}$$

gauge-invariant phase difference $\tilde{\phi} \equiv \phi - \frac{2\pi}{\Phi_0}\int_2^1 \mathbf{A}\cdot d\mathbf{l} = 2n\pi - 2\pi\frac{\Phi}{\Phi_0} = 2\pi(n - f)$

Symmetry and Phase Transition

Spacetime: homogeneous and isotropic

Symmetry of Physical Law: e.g. 뉴턴의 운동 법칙 $\mathbf{a} = \mathbf{F}/m$

Invariance under symmetry transformation

나란히 옮김translation, 돌림rotation, 시간 진행time translation

전하켤레charge conjugation, 홀짝성parity, 시간 되짚기time reversal

맞바꿈exchange/permutation

게이지gauge

응집물질condensed matter: 대칭성이 절로 깨질 수 있음
spontaneous symmetry breaking

→ 정돈(질서) order

Hamiltonian $H = H_0 + V$

Partition function $Z \equiv \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\beta H_0} e^{-\beta V}$

high-temp. limit: $\beta V \to 0$ noninteracting system

low-temp. limit: $\beta V \to \infty$ ground state

$T \to \infty$ noninteracting system (ideal) gas

$\downarrow$ perturbation expansion: density exp. (no λ-exp.)

What is in between? phase transition

$\uparrow$ elementary excitations: collective modes

$T \to 0$ ground state (solid crystal)

바닥상태 (= 진공)의 대칭성 깨짐 → 물질의 대칭성 깨짐

Water and Ice: H_2O 분자들의 집단

T

water (symmetry: translation, rot) disorder

entropy

T_c ← phase transition (symmetry breaking)

ice (broken symmetry) order

energy

0

Cooperativity among many constituents → emergent property

액체-고체, 자석, 초전도, 초기 우주, 기억 작용, DNA 풀어짐, 세포 분화, 피의 산소운반, 효소 작용, 여론 형성, 지각 작용, 도시 형성, 경기 변동과 공황, …

Order Parameter

How to specify the broken-symmetry state?

$\rightarrow$ order parameter

$$\psi \begin{cases} = 0 & \text{sym. state (disordered)} \\ \neq 0 & \text{unsym. state (ordered)} \end{cases}$$

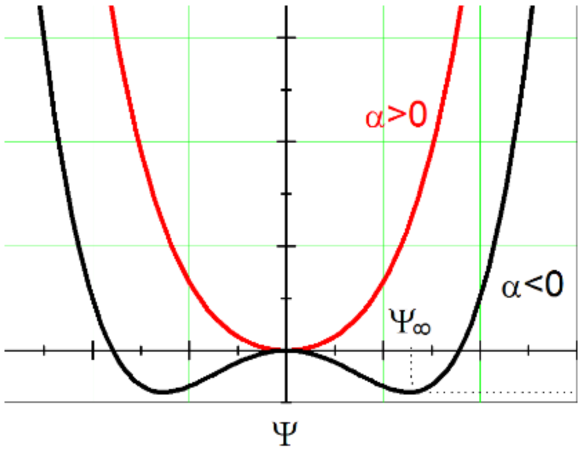
Free energy functional expanded in powers of ψ Landau theory

$$F(\psi) = F_0 + \alpha |\psi|^2 + b|\psi|^4 + \cdots$$

$$\rightarrow F_{\min}$$

$\Rightarrow$ equilibrium order parameter

$$\psi = \begin{cases} 0, & \alpha > 0 \\ \sqrt{\dfrac{-\alpha}{b}}, & \alpha < 0 \end{cases}$$



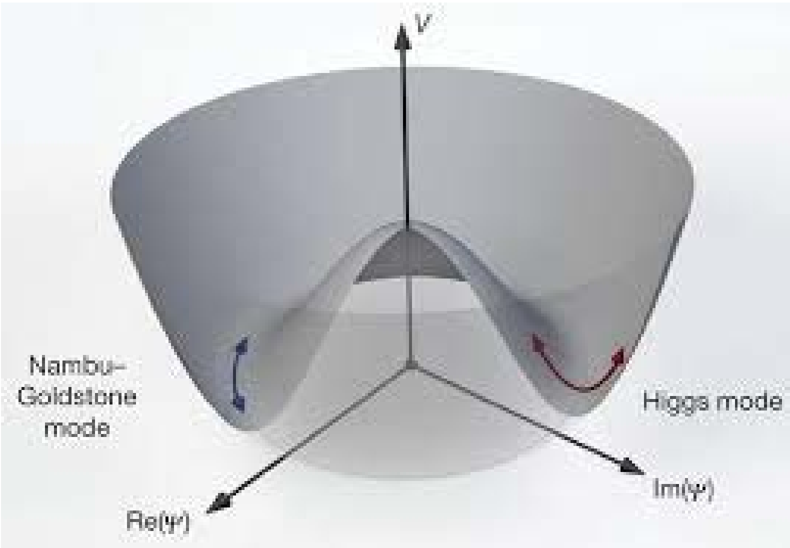
Usually ψ: continuous at transition, i.e., $\psi \rightarrow 0$ as $T \rightarrow T_c$ 2nd-order tr.

cf. 1st-order (discontinuous) transition

Discrete vs Continuous Symmetry

Discrete sym.: ψ may be a scalar (real) variable.   Ising model

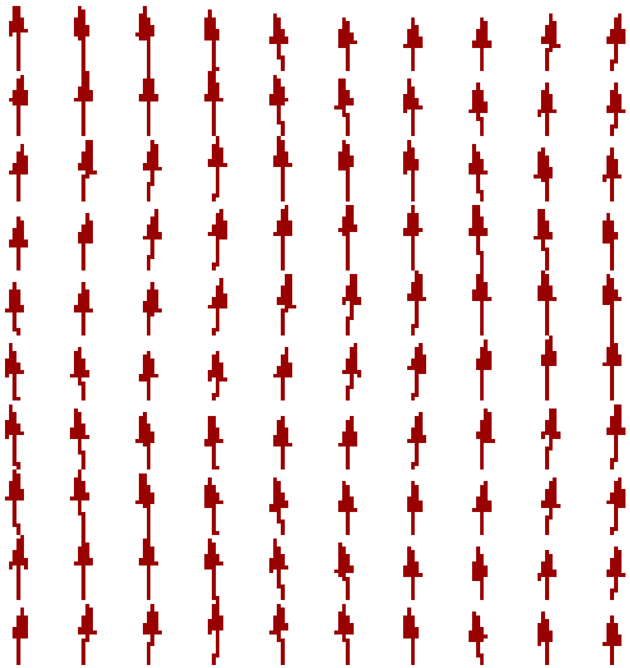
Continuous sym.: ψ has components, *phase angle*.   XY model



Goldstone theorem: continuous symmetry broken → Goldstone mode (massless)

Mermin-Wagner theorem: Continuous symmetry may not be broken in 2D.

Spin wave excitations



Order parameter $\psi = 0$: no long-range order
$\leftarrow$ no broken (continuous) symmetry

Correlation function

$$\Gamma(\mathbf{r}) \equiv \left\langle e^{i\phi(\mathbf{r})} e^{-i\phi(0)} \right\rangle$$

In the limit $r \to \infty$,

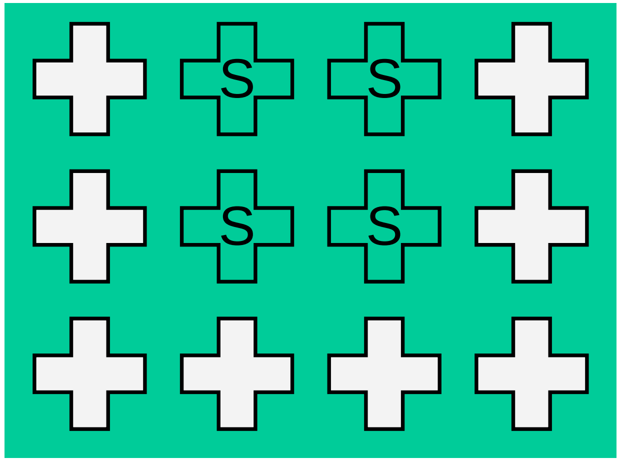
Spin wave excitations $\rightarrow$

$$\Gamma(r) \sim \begin{cases} \Gamma_\infty \ (\neq 0), & \text{LRO} \\ \mathrm{e}^{-r/\xi} & \text{disorder} \end{cases}$$

$\Gamma(r) \sim r^{-\eta}$ algebraic (QLRO) "critical"

Berezinskii-Kosterlitz-Thouless Transition

2D Superconducting Arrays



superconducting islands
weakly coupled by
Josephson junctions

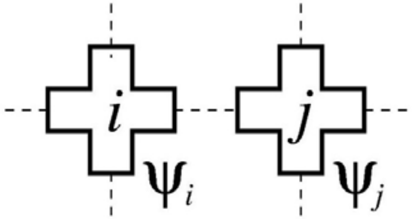
- Described by the 2D XY model
- Study of low-dim. physics
- Related to a variety of systems

  e.g. superconducting networks
  tight-binding electrons
  high-T_c superconductors
  quantum Hall system

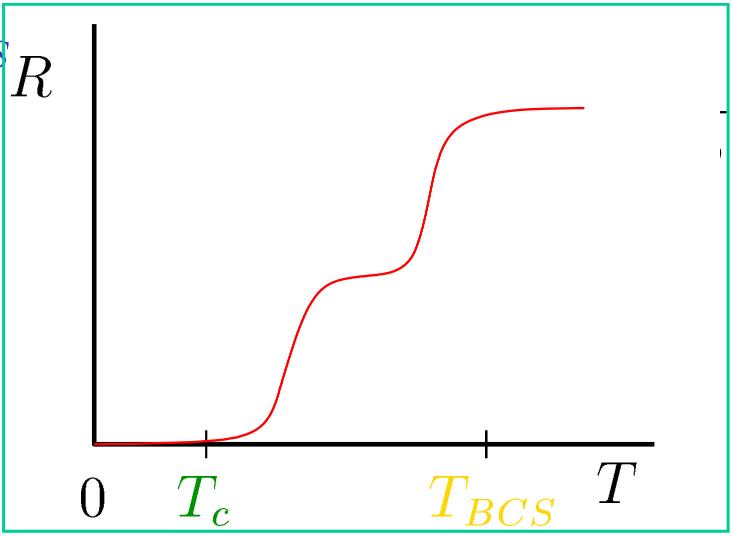
Ginzburg-Landau Description

- GL free energy

$$F = \sum_i (a|\Psi_i|^2 + b|\Psi_i|^4) + \sum_{\langle i,j \rangle} E_J |\Psi_i - \Psi_j|^2$$



- Two transition regions
  - $T = T_{BCS}$ $(a = 0)$ $R \neq 0$
  - $T = T_c$: $(\approx E_J)$ $R \to 0$

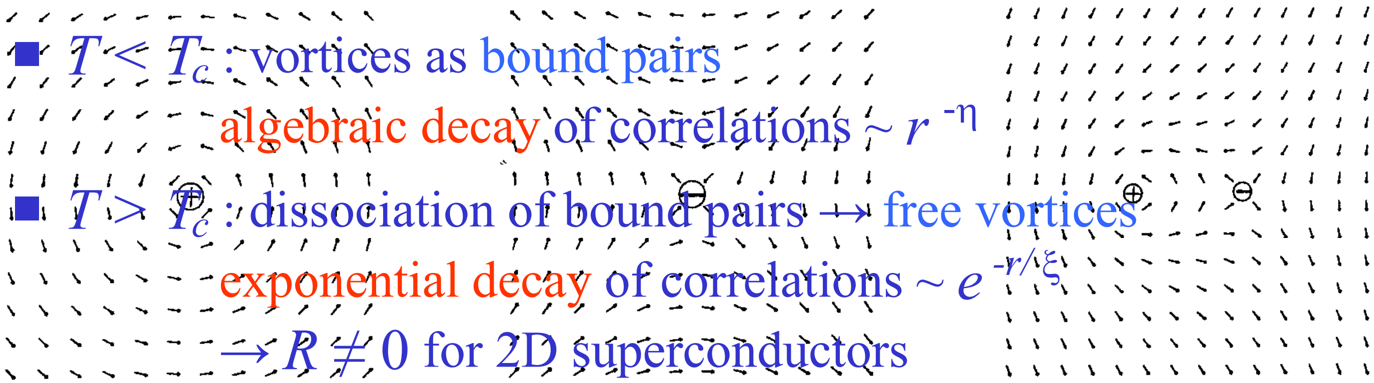


2D XY Model

- Lower transition region: phase-only approximation

$$\rightarrow H = -E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) \quad \text{: 2D XY model}$$

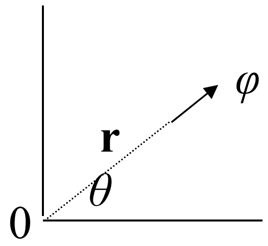
- Excitations $\begin{cases} \text{spin wave : Goldstone mode} \\ \text{vortex : topological defect} \end{cases}$



- $T < T_c$: vortices as bound pairs
  algebraic decay of correlations $\sim r^{-\eta}$
- $T > T_c$: dissociation of bound pairs $\rightarrow$ free vortices
  exponential decay of correlations $\sim e^{-r/\xi}$
  $\rightarrow R \neq 0$ for 2D superconductors
- BKT transition at $T = T_c$

(a) (b) (c)

vortex at $\mathbf{r} = 0$: $\varphi(\mathbf{r}) = \theta(\mathbf{r})$



$$\nabla\phi = \frac{1}{r}\hat{\boldsymbol{\theta}}$$

$$c\,|\nabla\psi|^2 = c\,|\psi|^2\,(\nabla\phi)^2 = \frac{1}{2}E_J\frac{1}{r^2}$$

energy of a single vortex

$$E_{1v} = \frac{1}{2}E_J\int d^2r\,(\nabla\phi)^2 = \frac{1}{2}E_J\int d^2r\,\frac{1}{r^2} = \pi E_J\ln\frac{R}{\xi}\xrightarrow{R\to\infty}\infty$$

free energy change asso. with free vortex formation

$$\Delta F = \Delta E - T\Delta S = \pi E_J\ln\frac{R}{\xi} - kT\ln\Omega = (\pi E_J - 2kT)\ln\frac{R}{\xi}$$

$T < T_c \equiv \pi E_J/2k$: bound pairs $\rightarrow$ algebraic decay

$T > T_c$: free vortices $\rightarrow$ exponential decay

$T = T_c$: ionization of vortices $\rightarrow$ BKT transition

topological (no sym. breaking)

Frustrated XY Model

$$H = -E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{ij})$$

$$A_{ij} \equiv \frac{2e}{\hbar c} \int_i^j \mathbf{A} \cdot d\mathbf{l}, \quad \sum_P A_{ij} = 2\pi\Phi/\Phi_0 \equiv 2\pi f$$

(f : gauge-invariant frustration)

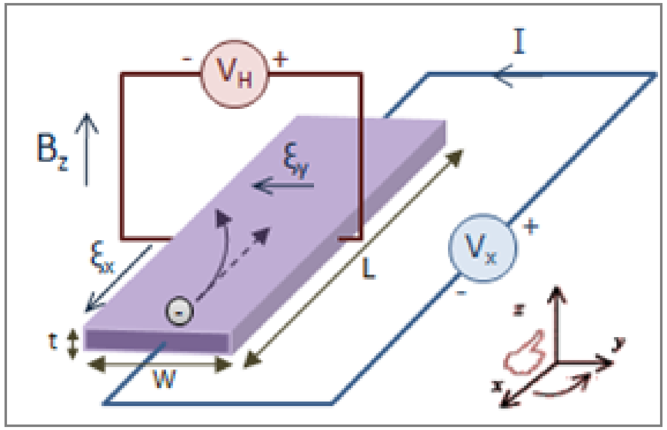
- Only frustration effects ($\leftrightarrow$ spin-glass)
  enter in a controllable way ($\leftarrow$ magnetic field) complex systems
- Discrete symmetry Z_q in addition to continuous U(1) symmetry
  $\rightarrow$ possibility of LRO in 2D (vortex + domain wall)
- Duality transformation $\rightarrow$ Coulomb gas of (fractional) charges

$$H = 2\pi^2 E_J \sum_{\langle \mathbf{R},\mathbf{R}' \rangle} [n_{\mathbf{R}} - f] G(\mathbf{R},\mathbf{R}') [n_{\mathbf{R}'} - f]$$

$f = 0$ (XY model): renormalization group analysis (Kosterlitz) $\rightarrow$ BKT transition

Quantum Hall Effect

Hall Effect



$$\mathbf{F} = -e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = 0$$

$$\Rightarrow cE_y - v_x B_z = 0$$

Hall voltage

$$V_H = E_y w = w \frac{v_x}{c} B_z$$

Current density $\quad j_x = nev_x = \sigma_{xy} E_y$

Hall coefficient $\quad R_H \equiv \frac{E_y}{j_x B_z} = \frac{1}{nec}$

Hall resistivity/conductivity $\quad \rho_{xy} = \frac{1}{\sigma_{xy}} = \frac{E_y}{j_x} = \frac{B}{nec}$

Integer Quantum Hall Effect

2D electrons in magnetic fields → Landau levels, each with degeneracy $g = \dfrac{\Phi}{\Phi_0} \equiv N_\phi$

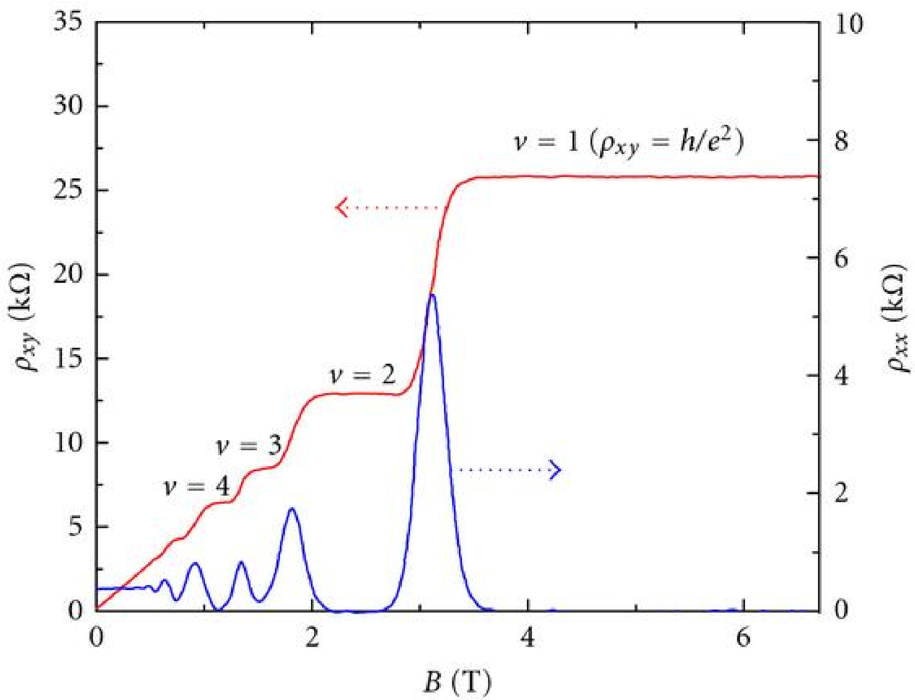
Hall conductivity

$$\sigma_{xy} = \frac{nec}{B} = \frac{nL_x L_y ec}{\Phi} = \left(\frac{N_e}{N_\phi}\right)\frac{ec}{\Phi_0}$$

$$= \nu \frac{e^2}{h} \quad (\nu \equiv N_e / N_\phi, \text{ filling factor})$$

accuracy ≤ 10^{-10}

Standard for electrical resistance:
Resistance quantum:
$R_K \equiv h/e^2 = 25812.807557\ \Omega$



impurities → level broadening
→ extended (center) + localized states (edges)
→ finite step width

Linear response theory

$$\sigma_{xy} = \frac{ie^2\hbar}{L^2}\sum_{n>0}\frac{V_{0n}^x V_{n0}^y - V_{0n}^y V_{n0}^x}{(E_n - E_0)^2} \quad \text{where } \mathbf{V} \equiv \sum_i \mathbf{v}_i \text{ and } V_{n0}^x \equiv \langle \psi_n | V_x | \psi_0 \rangle = \frac{L_x}{\hbar}\left\langle \psi_n \left| \frac{\partial H}{\partial \theta_x} \right| \psi_0 \right\rangle, \text{ etc.}$$

Under magnetic translation $T(\mathbf{l}) \equiv \prod_i^{N_e} e^{-(i/\hbar)\mathbf{\Pi}_i\cdot\mathbf{l}}$ where $\mathbf{\Pi} \equiv \mathbf{p} - \frac{e}{c}(\mathbf{A} - \mathbf{B}\times\mathbf{r})$, we have $T(L_x\hat{\mathbf{x}})|\psi\rangle = e^{-i\theta_x}|\psi\rangle$, etc.

$$\Rightarrow \sigma_{xy} = \frac{e^2}{h}\int\frac{\mathrm{d}^2\theta}{2\pi i}\left[\left\langle \frac{\partial\psi_0}{\partial\theta_x}\middle|\frac{\partial\psi_0}{\partial\theta_y}\right\rangle - \left\langle \frac{\partial\psi_0}{\partial\theta_y}\middle|\frac{\partial\psi_0}{\partial\theta_x}\right\rangle\right] = \frac{e^2}{h}C_1 \qquad \text{Chern number}$$

$$C_1 = \iint\frac{\mathrm{d}^2\theta}{2\pi}\left[\partial_{\theta_x}A_y - \partial_{\theta_y}A_x\right] = \frac{1}{2\pi}\oint \mathrm{d}\boldsymbol{\theta}\cdot\mathbf{A}, \quad \text{where } A_x \equiv i\left\langle \psi_0 \middle| \frac{\partial\psi_0}{\partial\theta_x}\right\rangle \qquad \text{Berry connection}$$

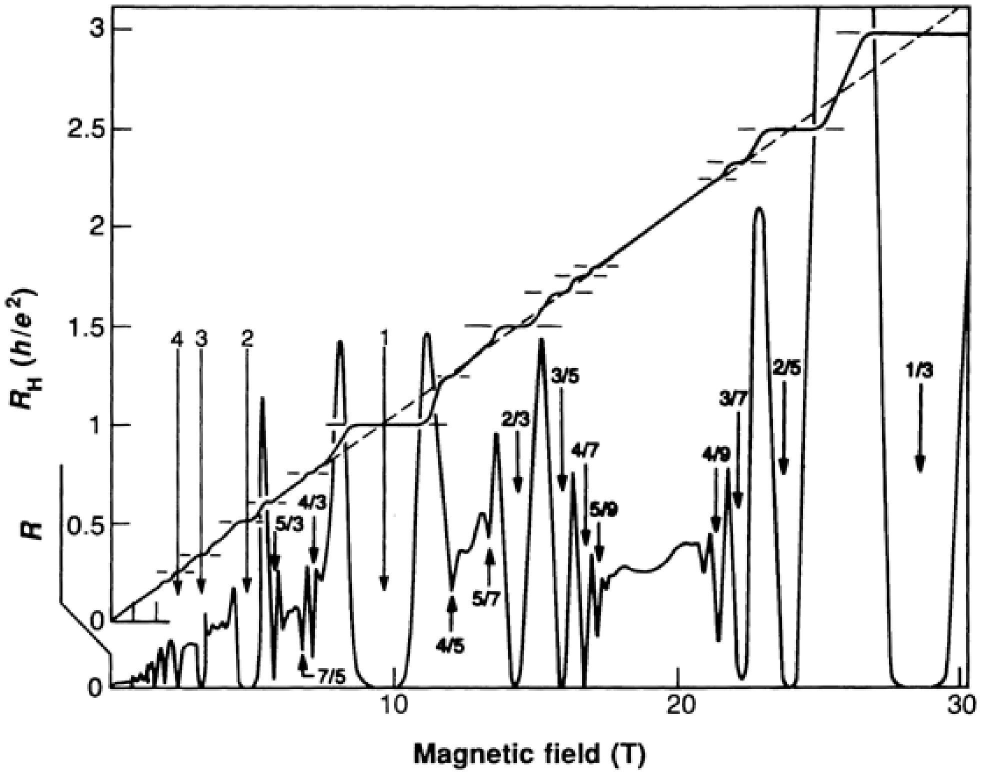
$$C_1 = \frac{1}{2\pi}\oint \mathrm{d}\boldsymbol{\theta}\cdot\nabla_\theta\Lambda = n \neq 0 \text{ if } \exists \text{ vortex (topological singularity) in the } (\theta_x, \theta_y) \text{ plane.}$$

Hall conductivity

$$\Rightarrow \sigma_H = \frac{e^2}{h}C_1 = n\frac{e^2}{h} \qquad \text{regardless of impurities and/or interactions}$$

integer quantum Hall effect (IQHE)

Fractional Quantum Hall Effect



$\sigma_H = \nu \dfrac{e^2}{h}$ with the filling factor $\nu = \dfrac{p}{q}$

IQHE regardless of impurities and interactions

Then why FQHE?

Condition for IQHE

1. Energy gap bet. ground & excited states
   incompressible fluid
2. Nondegenerate ground state:
   single-valued wave function on S^2

$\Rightarrow$ As $\theta = 0$ to 2π, we have $|\psi_0\rangle$ to $|\psi_0\rangle$ (up to a phase)

In FQHE ($\nu = p/q$):

1. Still valid (Fermi gap $\leftarrow \sigma_{xx} = 0$)
   Coulomb interactions $\rightarrow$ incompressibility
2. Multi-valued wave function on S^2
   q-fold topological degeneracy $|\psi_\alpha\rangle$, $\alpha = 1, 2, \ldots, q$

Degeneracy on a torus $L_x \times L_y$

Magnetic translation $T\left(\frac{L_x}{N_\phi}\hat{\mathbf{x}}\right) \equiv T_x$ etc., where $T(\mathbf{l}) \equiv \prod_i^{N_e} e^{-(i/\hbar)\mathbf{\Pi}_i\cdot\mathbf{l}}$ $\Rightarrow$ $T_x T_y = e^{-i2\pi p/q} T_y T_x$

$\Rightarrow$ q-fold degenerate eigenstates of H and T_y if $[H, T_{x/y}] = 0$ (clean sample)

$$|\psi_\alpha\rangle \equiv T_x^\alpha |\psi_0\rangle \quad (\alpha = 1,\ldots,q), \text{ where } T_y|\psi_0\rangle = e^{-i\theta_y}|\psi_0\rangle \text{ eigenstate of unitary op. } T_y$$

$$\Rightarrow T_y|\psi_\alpha\rangle = T_y T_x^\alpha|\psi_0\rangle = e^{-i2\pi\alpha p/q} T_x^\alpha T_y|\psi_0\rangle = e^{-i(\theta_y - 2\pi\alpha p/q)}|\psi_\alpha\rangle, \text{ also eigenstates of } T_y$$

$$\Rightarrow T(L_y\hat{\mathbf{y}})|\psi_\alpha\rangle = e^{-i[\theta_y(\alpha=0)-2\pi\alpha]}|\psi_\alpha\rangle, \text{ i.e., } |\psi_\alpha\rangle = |\psi_0;\theta_x,\theta_y + 2\pi\alpha\rangle,$$ q-fold degenerate eigenstates

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{q}\sum_{\alpha=1}^{q}\int\frac{\mathrm{d}^2\theta}{2\pi i}\left[\left\langle\frac{\partial\psi_\alpha}{\partial\theta_x}\middle|\frac{\partial\psi_\alpha}{\partial\theta_y}\right\rangle - \left\langle\frac{\partial\psi_\alpha}{\partial\theta_y}\middle|\frac{\partial\psi_\alpha}{\partial\theta_x}\right\rangle\right] = \frac{e^2}{h}\frac{1}{q}\int_0^{2\pi}\frac{\mathrm{d}\theta_x}{2\pi i}\int_0^{2\pi q}\mathrm{d}\theta_y\left[\left\langle\frac{\partial\psi_0}{\partial\theta_x}\middle|\frac{\partial\psi_0}{\partial\theta_y}\right\rangle - \left\langle\frac{\partial\psi_0}{\partial\theta_y}\middle|\frac{\partial\psi_0}{\partial\theta_x}\right\rangle\right]$$

$$= \frac{e^2}{h}\frac{1}{q}C_1$$ ←topological invariance from non-interacting $\sigma_H = \nu\frac{e^2}{h} = \frac{p}{q}\frac{e^2}{h}$ to interacting system

$$= \frac{p}{q}\frac{e^2}{h} \text{ for } \nu = \frac{p}{q}$$ plateau near $\nu \approx p/q$ ← localization of quasi-particles
(fractional charge/statistics)

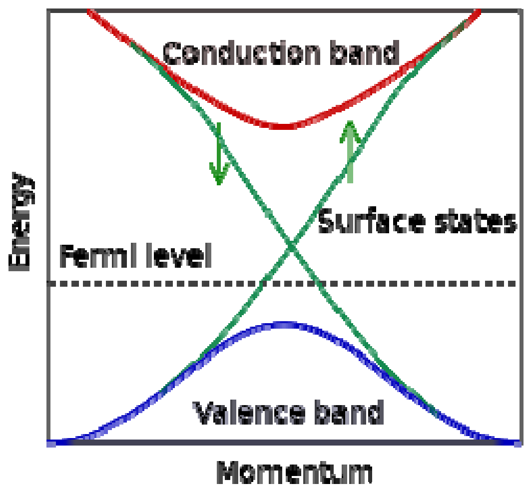
Other Topological Phases

F. D. M. Haldane; C.L. Kane and E.J. Mele

Quantum spin Hall effect and 2D topological insulators

spin up electron exhibits a chiral integer quantum Hall effect while the spin down electron exhibits an anti-chiral integer quantum Hall effect.

Surface (conducting) states of topological insulators are symmetry protected by charge conservation and time reversal symmetry.



Vortex Dynamics

Macroscopic Quantum Phenomena

$$H = 2e^2 \sum_{\langle i,j \rangle} [n_i - q] C_{ij}^{-1} [n_j - q] - E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{ij})$$

- Cooper pairs on the i^{th} grain
  charge $2en_i$ (charging energy) $\leftrightarrow$ phase ϕ_i (Josephson energy)
  canonical quantization $[n_i, \phi_j] = -i\delta_{ij}$ $\Rightarrow$ MQP
- Magnetic field **B** $\longrightarrow$ gauge field $A_{ij} \longrightarrow$ magnetic frustration f
- Gate voltage (electric field) $V \longrightarrow$ gauge charge $Q \longrightarrow$ charge frustration $q = Q/2e$

thermal fluctuations $\leftrightarrow$ quantum fluctuations
magnetic frustration $\leftrightarrow$ charge frustration

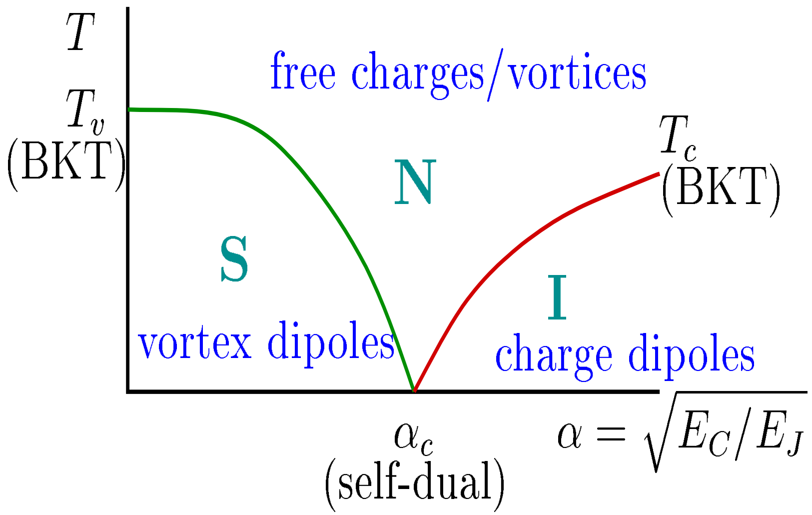
- Quantum phase transition (S-I) at $T = 0$ $\leftarrow$ quantum fluctuations

  $E_C \,(\equiv 4e^2/C) \ll E_J$: phase ordering $\longrightarrow$ superconductor

  $E_C \gg E_J$: charge ordering $\longrightarrow$ (Mott) insulator

$T > 0$ ($f = q = 0$): BKT transition

vortex unbinding (S-N) at T_v
charge unbinding (S-I) at T_c



Topological Quantization

- voltage $\longrightarrow$ charge motion $\longrightarrow$ current
  current $\longrightarrow$ vortex motion $\longrightarrow$ voltage
- $E_C \ll E_J$: ac driving $\longrightarrow$ voltage quantization

  $$f = \frac{r}{s}: \quad \langle V \rangle = \frac{n}{s}\frac{L\hbar\Omega}{2e}$$ GSS (Giant Shapiro steps)
- $E_C \gg E_J$: ac driving $\longrightarrow$ current quantization (Bloch osc.)

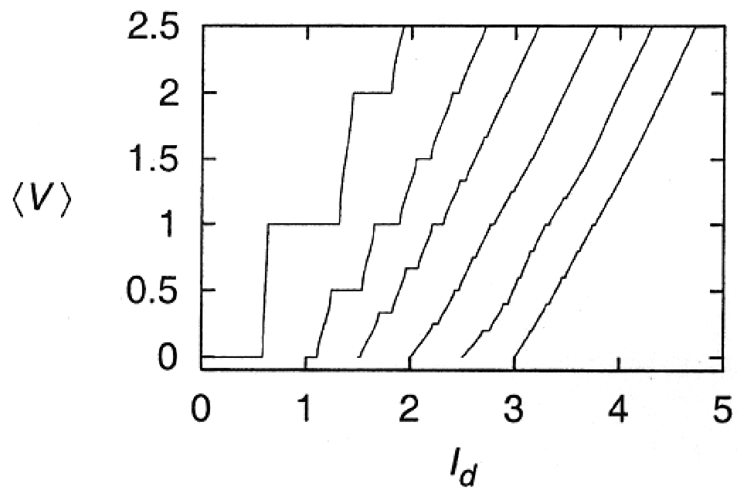
  $$q = \frac{r}{s}: \quad \langle I \rangle = \frac{n}{s}\frac{L2e\Omega}{2\pi}$$ GISS
- $E_C \sim E_J$: E_C/E_J provides K.E. of vortices/charges
  lattice structure destroyed $\longrightarrow$ quantum fluid
  $\longrightarrow$ conductance quantization
  quantum vortex: boson ($\longleftarrow$ Berry's phase) with hard core
  $\longrightarrow$ J-W transformation $\longrightarrow$ fermion with gauge field

  $$\rightarrow \quad \sigma_{xy} = m\frac{4e^2}{h} \quad (m = \text{even})$$ QHE

Mode Locking, Melting, and Transitions

ac+dc driving $I = I_d + I_a \cos \Omega t$ at $T = 0$

$\rightarrow$ voltage quantization: giant Shapiro steps (GSS)



$$f = 0: \quad \langle V \rangle = n \frac{L\hbar\Omega}{2e} \quad \text{IGSS}$$

$$f = r/s: \langle V \rangle = \frac{n}{s} \frac{L\hbar\Omega}{2e} \quad \text{FGSS}$$

cf. devil's staircase

- mode locking $\leftarrow$ topological invariance
- chaos

Non-simply Connected Geometry

- Interference effects

  charge moving in magnetic field: Aharonov-Bohm effect

  $\rightarrow$ persistent current $\quad I = \dfrac{2e}{h}\dfrac{\partial E}{\partial f}$

  vortex moving in gauge charge field: Aharonov-Casher effect

  $\rightarrow$ persistent voltage $\quad V = \dfrac{1}{2e}\dfrac{\partial E}{\partial q}$

- Coupled Array
  - charge transport via excitons

    (pairs of excess and deficit Cooper pairs)
  - interesting phase transition and transport properties

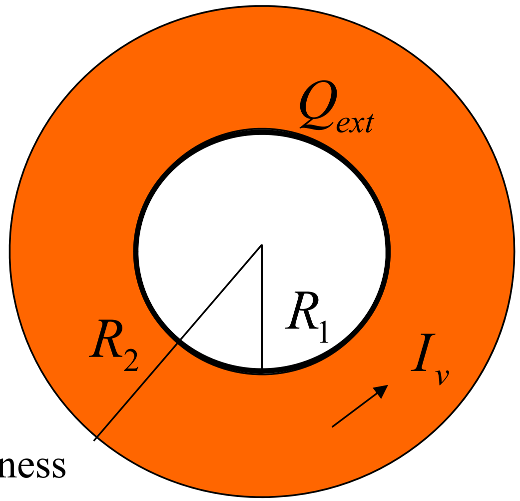
Vortex Current

Annulus-shaped array with induced charge $Q_{ext} \equiv -2eq$ on the inner boundary

- N vortices ➡ Laughlin-type wave function
- persistent vortex current $I_v = \dfrac{c}{2e}\dfrac{\partial E}{\partial q}$
- persistent voltage $V = -\dfrac{I_v}{c}$

$\Rightarrow$ spontaneous voltage

$$V_s = \frac{NE_c}{\pi^2 e}\left[1 + \frac{aNE_c}{4\pi^2 e^2 \ln(R_2/R_1)}\right]^{-1}$$



a: thickness

Epilogue

Explanation of the Nobel Prize in Physics 2016

David J. Thouless, J. Michael Kosterlitz, and F. Duncan M. Haldane

Why me?

1. Research Experiences
2. Personal Relations

Phase transition: BKT transition, frustrated XY model

Localization: localized and critical states

Spin-glass: gauge-glass

Quantum Hall effect: boson system

Spectrum of the Schrödinger operator: incommensurate system

Vortex dynamics: quantum vortex

M.Y. Choi and S. Doniach, “Phase transitions in uniformly frustrated XY models”, Phys. Rev. B **31**, 4516 (1985).

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Y.-J. Kim, M.Y. Choi, and M.H. Lee, “Anomaly and singular continuous spectrum in a one-dimensional incommensurate system”, J. Phys. A **22**, L571 (1989).

M.Y. Choi, “Topological invariance of superconducting arrays”, Phys. Rev. B **46**, 564 (1992).

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J. Yi, M.Y. Choi, K. Park, and E.-H. Lee, “Peierls gap in a mesoscopic ring threaded by a magnetic flux”, Phys. Rev. Lett. **78**, 3523 (1997).

M.-S. Choi, M.Y. Choi, T. Choi, and S.-I. Lee, “Cotunneling transport and quantum phase transitions in Josephson-junction chains with charge frustration”, Phys. Rev. Lett. **81**, 4240 (1998).

M.Y. Choi and S.Y. Park, “Novel transition in the two-dimensional gauge glass”, Phys. Rev. B **60**, 4070 (1999).

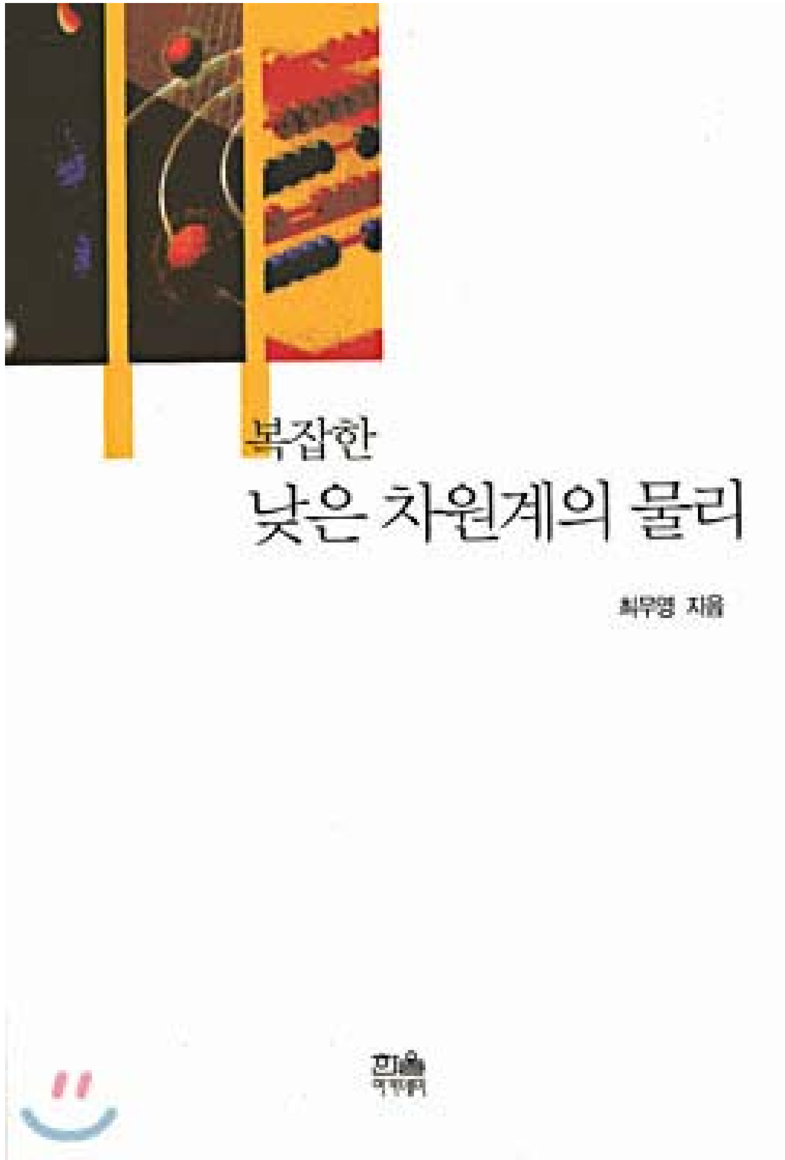
S.Y. Park, M.Y. Choi, B.J. Kim, G.S. Jeon, and J.S. Chung, “Intrinsic finite-size effects in the two-dimensional XY model with irrational frustration”, Phys. Rev. Lett. **85**, 3484 (2000).

M.Y. Choi and J. Choi, “Topological quantization and degeneracy in Josephson-junction arrays”, Phys. Rev. B **63**, 212503 (2001).

M.Y. Choi and D.J. Thouless, “Topological interpretation of subharmonic mode locking in coupled oscillators with inertia”, Phys. Rev. B **64**, 014305 (2001).

M. Lee and M.Y. Choi, “Correspondences and quantum description of Aharonov-Bohm and Aharonov-Casher effects”, J. Phys. A **37**, 973 (2004).

For details of QHE and other topological phenomena in low dimensions, see the book:

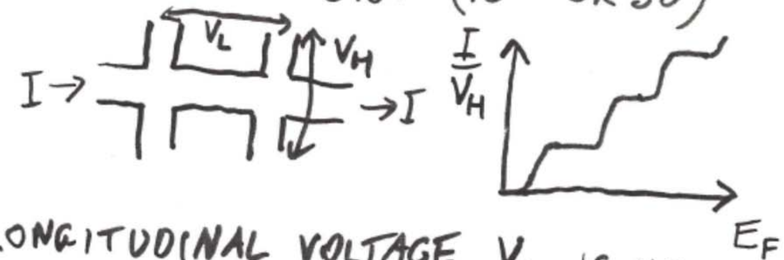


2. Personal Relations

INTEGER QUANTUM HALL EFFECT

(V. KLITZING '80)
DORDA, PEPPER

TWO DIMENSIONAL ELECTRONGAS IN STRONG MAGNETIC FIELD, LOW TEMPERATURE HAS HALL CONDUCTANCE ne^2/h TO VERY HIGH PRECISION (10^{-9} OR SO)



LONGITUDINAL VOLTAGE V_L IS VERY SMALL AT PLATEAUS OF V_H.

FRACTIONAL QUANTUM HALL EFFECT

(STÖRMER, TSUI, GOSSARD '82)

FOR HIGH MOBILITY SAMPLES, PLATEAUS ARE FOUND AT $\nu e^2/h$, WHERE ν IS SIMPLE FRACTION (USUALLY DENOMINATOR IS ODD)

"Without experimental realization, it's not physics but merely an intellectual game." D.J. Thouless

"Is theory of everything theory of anything?" P.W. Anderson

Betteridge's law or Hinchliffe's rule

"Is Hinchliffe's rule true?" liar paradox

Theory of Everything

Theory of Something Real

or

Theory of Nothing

symmetry and breaking, topological order, gauge field, Anderson-Higgs mechanism, fractional charge, fractional statistics (anyon), Berry (geometric) phase and BAC effect, topological quantum number, … emergence

고맙습니다.