Classical Mechanics, Qualifying Exam in Summer, 2006

1. A point mass m moves on the parabolic surface given by $z = \rho^2$. The parabolic surface is located under the uniform Earth gravity g in $-z$ direction.

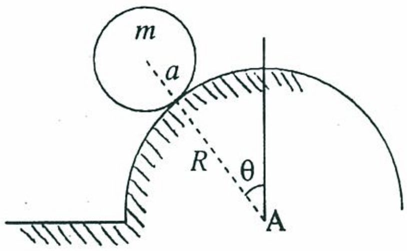
(a) Write the Lagrangian in the cylindrical coordinates (ρ, ϕ, z).

(b) Show that the angular momentum l about the z axis is conserved.

(c) For a given energy E, find the lower and upper bounds (turning points) of ρ. Assume that the zero of the potential energy is at $z = 0$.

(d) By use of Lagrange multipliers, find the force of constraint.

2. We put a solid sphere (of radius a, mass m, and rotational inertia $\frac{2}{5}ma^2$) on top of a big hemisphere of radius R fixed on the ground as shown in the figure. The solid sphere began to roll without slipping from the top with practically zero initial velocity. The gravitational acceleration is g and the angular velocity of the center of the mass of the solid sphere with respect to the point A is $\omega(= \frac{d\theta}{dt})$.



In this case, the angular velocity of the solid sphere with respect to the axis passing through its own center of mass (and perpendicular to the plane of the figure) is given by $\frac{R+a}{a}\omega$.

(a) Express the total mechanical energy (kinetic plus potential) at the position θ in terms of the parameters given above. Then express ω as a function of θ making use of the conservation of the mechanical energy. The zero of the potential energy is taken to be at the ground ($\theta = \frac{\pi}{2}$).

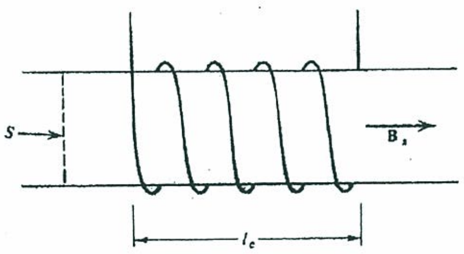
(b) Derive the normal force N exerted on the solid sphere from the big hemisphere at the position θ. Make use of the fact that the center of mass of the solid sphere is in circular motion with respect to A.

(c) What is the magnitude of torque $\vec{\tau}$ exerted on the solid sphere at the position θ?

(d) Suppose that the maximum coefficient of static friction is μ_s. If θ exceeds a certain critical angle θ_c, the solid sphere cannot maintain the rolling without slipping and begins to slip. Derive the relation which the critical angle θ_c satisfies. (You don't have to solve the relation.)

Electrodynamics, Qualifying Exam in Summer, 2006

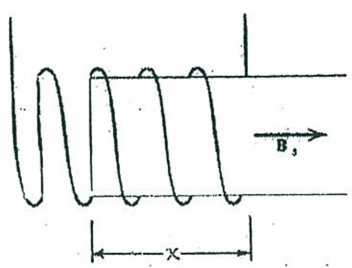
1. Consider two solenoids, one a long solenoid with turn density n_s and length l_s, another a rather short coil with turn density n_c and length l_c wound over the solenoid. Since the coil is wound over the solenoid the cross section S is assumed to be equal for both solenoid and coil. Permeability in vacuum is μ_o.



a) If there is a current I_s through the solenoid, calculate B filed inside of the solenoid.

b) Find out the self inductance of solenoid (Ls), coil (Lc), and mutual inductance (M_{sc}) between solenoid and coil.

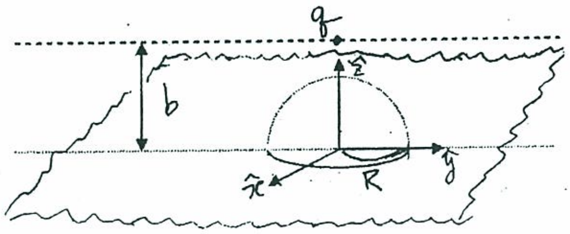
Now the coil is placed near the end of the first solenoid as shown in the figure, thus the length of overlapping region is x. Here you can ignore the end effect of solenoid and coil. Now solenoid and coil carry current I_s and I_c.



c) Find out the total energy of the system.

d) What is the magnitude of the force between the solenoid and coil?
Is the force attractive or repulsive?

2. Let's consider an infinite conducting plane. A conducting hemisphere of radius R is placed at the center position as shown in the figure. We would like to understand electric characters in this geometry when an external charge is introduced.

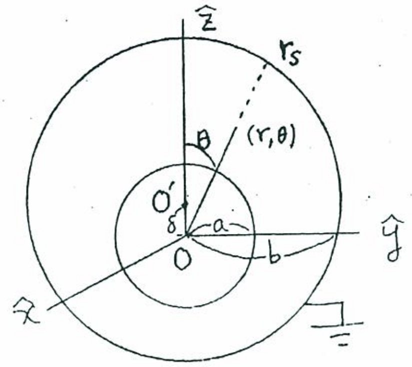


a) If a point charge q is placed at (0, 0, b) above the center of sphere, find all image charges and their positions. (here b>R)
Hint: First consider a conducting sphere only, to find an image charge. Then add a conducting plane.
b) From the result of (a), find the magnitude and the direction of the total dipole moment of the system.
c) Now consider an electron traveling from y=-∞ to y=∞ along the line x=0 and z=b. In this case, the dipole moment will vary depending on the position of the electron. If there are n electrons passing in equal intervals per second, what would be the major angular frequency ω of the dipole? Assume that the fast electron travels nearly at the speed of light and neglect the deflection of the electron due to the conductor. (Hint: this is a simple problem that requires no calculation!).
d) The oscillating electric dipole moment can be written as $\vec{p} = p_{\max} \cos \omega t \, \hat{z}$ and it generates an EM wave. At large enough distance from the dipole, fields are given as

$$\vec{E} = \frac{k^2}{4\pi\varepsilon_o}[(\hat{n}\times\vec{p})\times\hat{n}]\frac{e^{ikr}}{r}$$

where k=ω/c . What is the electric field due to the radiation at a point on the y axis located far away from the center?

3. Consider an isolated metal sphere of radius a with charge q, which is placed inside of a grounded metal shell of radius b (>a). The center O of the inner sphere is at the origin (0, 0, 0), while that of outer shell O' is displaced by a small amount δ along z direction so that the center is (0, 0, δ). (See the figure.)



a) First assume that δ=0 (i.e., centers of two spheres coincide),

   Find the potential $\phi(r,\theta)$ at an arbitrary

point (r, Θ) in between the inner and outer spheres.

b) Now consider the case of small but nonzero δ. Express the point r_s on the surface of outer shell in terms of b, Θ, and δ. (only upto first order of δ)

c) Write down all the necessary conditions or equations that should be considered to

   determine the potential $\phi(r,\theta)$

   at any point in between two spherical conductors.

d) Now obtain the potential $\phi(r,\theta)$ that is valid

   to first order in δ.

e) Also find the charge density distribution over the surface of the inner sphere.

Quantum Mechanics, Qualifying Exam in Summer, 2006

1. Consider the one-dimensional problem of a particle confined by rigid walls to the region $-L/2 \le x \le L/2$. Suppose that there is a positive potential barrier in the interior of the region with the form $V(x) = \frac{\hbar^2 v}{2mL}\delta(x)$. Since it is separated into Region I ($-L/2 \le x < 0$) and Region II ($0 < x \le L/2$), there is no reason to use the same wave function in Region I and Region II.

(a) Show that there is parity symmetry.

(b) Obtain the discontinuity of $\frac{d\psi}{dx}$ at $x = 0$. i.e.,

Obtain $\left(\frac{d\psi}{dx}(0^+) - \frac{d\psi}{dx}(0^-)\right)$

(c) Obtain even parity eigenfunctions and eigenenergies.

(d) Obtain odd parity eigenfunctions and eigenenergies.

(e) Show that in the limit of large v the energy difference between the even and odd solutions become

$$\Delta E \equiv E_n^{odd} - E_n^{even} \cong \frac{(4\pi\hbar n)^2}{mvL^2},$$

where $n = 1,2,3,\ldots$

(f) Assume that v is large and that the particle is initially confined to the left half ($-L/2 \le x \le 0$) with the energy approximately E_n, and the wave function is given by.,

$$\psi(x, t=0) = \frac{1}{\sqrt{2}}\left(\psi_n^{even} - \psi_n^{odd}\right).$$

Show that the probability of the particle being in the right half ($0 \le x \le L/2$) at time $t(>0)$ is

$$P(t) \cong \frac{1}{2}\left(1 - \cos\left(\frac{\Delta E}{\hbar}t\right)\right)$$

2. Consider a particle moving in the central potential

$V(r) = \frac{A}{r^2} - \frac{B}{r}$ (A, B are positive number)

Energy $E > 0$

a) Show that the Schroedinger equation for the radial wave function R of the particle can be reduced ($\rho = 2r\frac{\sqrt{(-2mE)}}{\hbar}$) to

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[-\frac{1}{4} + \frac{\alpha}{\rho} - \frac{s(s+1)}{\rho^2}\right]R = 0.$$

Also derive the expressionsof α and $s(s+1)$, respectively.

(Hint: In case of central potential *V(r)*, Schroedinger equation for the radial wave function *R* is

$$-\frac{\hbar^2}{2mr}\frac{d^2}{dr^2}(rR) + \frac{\hbar^2 l(l+1)}{2mr^2}R + V(r)R = ER)$$

b) Find the asymptotic behaviors of R for <u>large ρ</u> and <u>small ρ</u>, respectively. Do not consider a diverging solution.

c) Show that wave function R in a) can be written as

$R(\rho) = \rho^s e^{-\rho/2} L_n^{2s+1}(\rho)$

where n = α - s - 1 is a non-negative integer and $L_n^{2s+1}(\rho)$ is an associate Laguerre polynomial.

(Hint: You can prove that *R(ρ)* satisfies the generalized associate Laguerre equation and explain why *n* should be an integer. The generalized associate Laguerre equation is $xy''(x) + (\nu + 1 - x)y'(x) + \lambda y(x) = 0$. If λ is non-negative integer, it can be reduced to associate Laguerre equation and its solution $L_\lambda^\nu(\rho)$ becomes a polynomial of finite number of terms.)

d) Find out the energy levels and prove that it can be reduced to that of hydrogen atom if A=0.

3. Consider the paramagnetic resonance of an electron. The Hamiltonian of an electron in the uniform magnetic field B is

$$H = -\vec{M} \bullet \vec{B} = \frac{\hbar egB}{4mc}\sigma_z = \hbar\omega_0\sigma_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Initially (t=0) the state of the electron is $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Obtain $\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$.

(b) Obtain the expectation value of S_x at time *t*, namely $\langle S_x(t) \rangle$. Here, $S_x \equiv \frac{\hbar}{2}\sigma_x$.

(c) Now there is a perturbation in the magnetic field as

$$\vec{B} = B\hat{z} + B_1\cos(\omega t)\hat{x}$$

Show that the Schrödinger equation is in the form

$$i\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \omega_0 & K \\ K & -\omega_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

and obtain *K*. [Here, $\omega_0 = \frac{egB}{4mc}$.]

(d) Let $a(t) = A(t)e^{-i\omega_0 t}$, $b(t) = B(t)e^{+i\omega_0 t}$.

Show that

$$i\frac{d}{dt}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

and obtain *P*.

(e) Show that $\quad P \cong \frac{1}{2}\omega_1 e^{i(2\omega_0 - \omega)t}$

if the situation is close to resonance. Here, $\omega_1 = \frac{egB_1}{4mc}$.

Statistical Mechanics, Qualifying Exam in Summer, 2006

1. We consider a classical ideal gas of N indistinguishable particles in a three-dimensional box of volume V. The system is described by Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m},$$

where $\boldsymbol{p}_i$ is a linear momentum of particle i and m is a particle mass. We assume that N and V are sufficiently large.

cf. Use following formulas, if necessary,

$$\begin{aligned}
\int_{-\infty}^{\infty} dx \exp(-x^2) &= \sqrt{\pi}, \\
\int_{0}^{\infty} dx x^{1/2} \exp\left(-\frac{x}{A}\right) &= \frac{\sqrt{\pi}}{2} A^{3/2} \quad (A > 0), \\
\int d^n\boldsymbol{x} f(x) &= \frac{2\pi^{n/2}}{(n/2-1)!} \int dx x^{n-1} f(x), \\
& \quad (\boldsymbol{x} : n\text{-dimensional vector}, \quad x \equiv |\boldsymbol{x}|) \\
\ln N! &\approx N \ln N - N \qquad \text{for large } N.
\end{aligned}$$

(a) When the gas is in equilibrium with heat reservoir at temperature T, obtain the Helmholtz free energy F, the entropy S, and the average energy U.

(b) When the ideal gas above is isolated with energy E, compute the entropy by counting the number of microscopic states.

(c) By defining temperature T in microcanonical ensemble, show the equivalence between the entropies which are obtained by microcanonical ensemble and canonical ensemble approaches.

(d) If the particles are spin-0 bosons, obtain the expression in an integral form for average number of bosons in grand canonical ensemble. Show that the distribution function reduces to classical Boltzmann distribution if the average distance between bosons are very large compared to thermal wavelength [$\lambda_T \equiv (h^2/2\pi m k_B T)^{1/2}$]. You may use the fact that the density of states is given by $N(\epsilon) = 2\pi V(2m)^{3/2}\epsilon^{1/2}/h^3$

2. We consider the system in which identical atoms are piled up on N sites. The sites are aligned in a line. It is impossible to pile up atoms on each site more than the atoms stacked on its left site, and the total energy increases by the amount ϵ whenever one atom is added on any site. The system is in equilibrim at temperature T.

(a) When at most one atom can be piled up on each site, obtain the partition function. Calculate the average number of atoms in the limit $N \to \infty$.

(b) In case that up to two atoms can be piled up on each site, calculate the average number of atoms in the limit $N \to \infty$.

(c) We reconsider the system where at most one atom can be piled up on each site. Instead of the constraint that we cannot pile up atoms more than those stacked on its left site, the system now has the condition that the total energy increases additionally by the amount V when the number of atoms on a site is larger than that on its left site. Write down the partition function. (You do not need to evaluate the sum.)