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과 목 명 : Quantum Mechanics 2010 . 12. 22 시 행

1. Consider a harmonic oscillator system with the Hamiltonian operator

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2),$$

where variables $\hat{q}$ and $\hat{p}$ satisfy the commutation relation $[\hat{q}, \hat{p}] = i$.

(a) For $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$, show that

$$\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2}.$$

Also show that allowed eigenvalues of $\hat{H}$ are $E_n = (n + \frac{1}{2}).(n = 0, 1, 2, \cdots)$ [5pts]

(b) Construct nth energy eigenstate $|n\rangle$ in terms of the ground state $|0\rangle$, $\hat{a}$ and $\hat{a}^\dagger$. (The states are normalized : $\langle 0|0\rangle = 1 = \langle n|n\rangle$) [10pts]

(c) Get an explicit expression for the ground state wave function $\psi_0(q) = \langle q|0\rangle$. [10pts]

(d) Obtain the expression for the translation operator $\hat{D}(z)$ which has the following properties. [10pts]

- $\hat{D}^\dagger(z)\hat{a}\hat{D}(z) = \hat{a} + z$,
- $\hat{D}^\dagger(z)\hat{D}(z) = 1$,

where z is a complex number.

(e) Construct $|z\rangle$ which is a normalized eigenstate of $\hat{a}$, i.e., $\hat{a}|z\rangle = z|z\rangle$. [5pts]

2. The spin operators for spin-1/2 particles are given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the S_z-eigenstate basis.

(a) Derive the eigenvalues and the normalized eigenstates for the S_x operator in terms of the S_z-eigenstate basis. [10pts]

(b) Suppose that sequential Stern-Gerlach (SG) experiments for spin-1/2 particles are performed as depicted in the figure below. The y-axis corresponds to the direction of the spin-1/2 particles emitted from the source (S). The first SG experiment is performed along the x-axis (i.e., S_x is measured), and "spin down" is blocked (i.e., only "spin up" is selected for the following experiment). The other SG experiment is then performed along the z-axis. Find the expectation value of the final spin S_z measurement. [10pts]

Now suppose that a composite system of an electron (e) and a positron (p) is in the state

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|\uparrow\rangle_e|\downarrow\rangle_p - \sqrt{\frac{1}{3}}|\downarrow\rangle_e|\uparrow\rangle_p,$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the spin-up and spin-down eigenstates of S_z, respectively. A SG experiment is performed along the z axis for the *electron* while another SG experiment is performed along the x axis for the *positron*.

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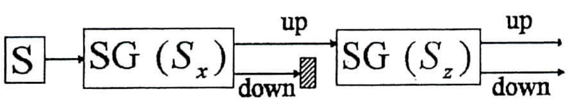
(c) What is the probability of getting "spin up" for the *electron* and "spin down" for the *positron*? [10pts]

(d) What is the expectation value of this composite measurement $S_z^{(e)} \otimes S_x^{(p)}$? [10pts]



3. The Hamiltonian for a relativistic particle in the presence of a central potential can be taken as $H = \sqrt{m^2c^4 + p^2c^2} + V(r)$ where $p^2 = \vec{p}\cdot\vec{p}$.

(a) If we wish to take into account relativistic effects perturbatively, show that the above Hamiltonian can be replaced by $H = H_0 + H'$ where

$$H_0 = \frac{p^2}{2m} + V(r)$$

and

$$H' = -\frac{(p^2)^2}{8m^3c^2}$$

up to $\mathcal{O}((\frac{p^2}{m^2c^2})^3)$. [10pts]

(b) If $|\psi\rangle$ is a normalized eigenstate of H_0 with eigenvalue E_0, show that it is possible to express its expectation value of H' as

$$\langle\psi|H'|\psi\rangle = -\frac{1}{2mc^2}[E_0^2 - 2E_0\langle V\rangle + \langle V^2\rangle]$$

where $\langle V\rangle \equiv \langle\psi|V|\psi\rangle$. [10pts]

(c) In the case of a hydrogen atom (i.e., $V(r) = -\frac{\alpha\hbar c}{r}$), the unperturbed ground state wave function is given by (here, $a = \frac{\hbar}{mc\alpha}$)

$$\psi_{100}(\vec{r}) = \frac{1}{\sqrt{\pi}} a^{-\frac{3}{2}} e^{-\frac{r}{a}}.$$

Use the result of (b) to find the first order (relativistic) correction $\Delta E^{(1)}_{n=1}$. (Ignore the spin-orbit couplings. For $n = 1$, $E_0 = -\frac{1}{2}mc^2\alpha^2$.) [10pts]

(d) For the $n = 2$ states of hydrogen atom, wave functions are given by

$$\psi_{2lm}(\vec{r}) = R_{2l}(r) Y_{lm}(\theta, \phi)$$

($m = \pm1, 0$ for $l = 1$, $m = 0$ for $l = 0$), where

$$R_{20}(r) = \frac{1}{\sqrt{2}} a^{-\frac{3}{2}} (1 - \frac{r}{2a}) e^{-\frac{r}{2a}},$$

$$R_{21}(r) = \frac{1}{\sqrt{24}} a^{-\frac{3}{2}} e^{-\frac{r}{2a}} (r/a).$$

What is the first order corrections $\Delta E^{(1)}$ for $(nlm) = (200), (211), (210), (2,1,-1)$ states? (For $n = 2$, $E_0 = -\frac{1}{8}mc^2\alpha^2$.) Also explain why it is unnecessary to use the degenerate perturbation theory here although the unperturbed $n = 2$ energy states are degenerate. [10pts]

(*Hint*) $\int_0^\infty x^n e^{-x} dx = n!$.

Statistical Physics

1. Let us consider N noninteracting molecules moving freely on a one dimensional line of length L at temperature T. Each molecule is composed of two distinguishable atoms, and is governed by the Hamiltonian

$$H(p_1, p_2, x_1, x_2) = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}K|x_1 - x_2|^2,$$

where $x_i, p_i (i = 1, 2)$ are the position and the momentum of the ith atom in the molecule. We assume that the molecules are indistinguishable classical particles, N and L are very large, and you can use $\ln N! \sim N \ln N - N$.

   (a) Find the Helmholtz free energy $A(T, L, N)$ of the system. [10pts]

   (b) Find the average energy U and show that the result is in agreement with the equipartition theorem. [10pts]

   (c) Find the average distance between two atoms in a molecule. [10pts]

   (*Hint*) $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$.

2. The ideal gas law (i.e., the ideal gas equation of state) cannot predict or explain many of the real gas phenomena including the gas-liquid phase transition, the presence of critial point, and the Joule expansion cooling. The van der Waals' gas model remedies much of the drawbacks. The van der Waals' equation of state, driven by applying the virial theorem to the gas of hard spheres (with weak, fairly short range, attractive forces), can be written as

$$(p + \frac{a}{V^2})(V - b) = RT$$

for a mole of gas. In the above equation, p is the pressure, V the molar volume, R the gas constant, and T is the (absolute) temperature of the gas.

   (a) Describe as accurately as possible the physical significance that the constants a and b have for a given gas. [7pts]

   (b) Find out the second virial coefficient B_2 of the van der Waals' gas in terms of the constants a and b, where the virial coefficients $B_n (n = 1, 2, \cdots)$ are defined by the equation

$$p = \frac{RT}{V} \sum_{n=1}^{\infty} B_n (\frac{1}{V})^{n-1}.$$

   [8pts]

   (c) At temperature T_t lower than the critical temperature T_c, the gas can be shown to exhibit a phase transition at pressure $p_t(T_t)$ where the jump in molar volume occurs. Show that the entropy change due to the transition is given by

$$\Delta S \equiv S_g - S_l = R\ln(\frac{V_g - b}{V_l - b}),$$

where S_g, S_l, V_g and V_l are the molar entropies and molar volumes for the gas and liquid phases, respectively. [8pts]

(d) Show that the Joule coefficient of the gas, when it is freely expanding (i.e., expansion at constant internal energy), is given by

$$\mu_J \equiv (\frac{\partial T}{\partial V})_U = -\frac{a}{C_V V^2},$$

where C_V is the molar heat capacity of the gas at constant volume. [7pts]

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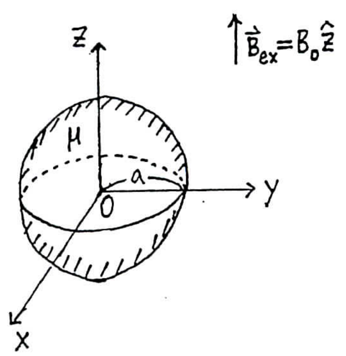
과 목 명 : Electrodynamics 2010 . 12. 22 시 행

1. Let us consider a sphere of radius a with a linear magnetic permeability μ, which is placed in a uniform external magnetic field $\vec{B}_{\text{ex}} = B_0 \hat{z}$.



(a) Show that magnetic scalar potential Φ_M (i.e., $\vec{H} = -\nabla\Phi_M$) can be defined, and that Φ_M satisfies Laplace's equation inside and also outside the sphere. [8pts]

(b) Find Φ_M everywhere. [12pts]

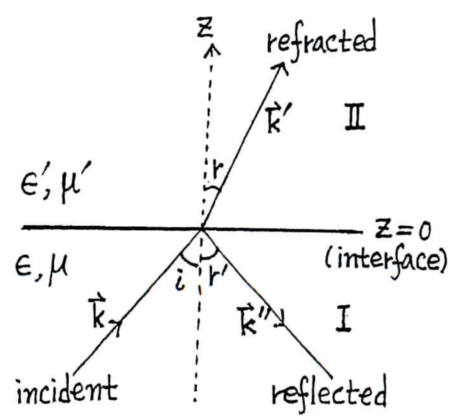
(c) Evaluate the magnetization $\vec{M}$ inside the sphere. [10pt]

(d) For the magnetization found in (c), obtain the effecitve current density inside and on the surface of the sphere. [10pts]

2. We wish to understand the behavior of electromagnetic waves across two different media (which are nonconducting), based on macroscopic Maxwell equations.

(a) First give the expressions for the electric and magnetic fields which represent a monochromatic plane wave propagating in a homogeneous medium (characterized by dielectric constant ϵ and magnetic permeability μ), with brief explanation for having such structure. [10pts]

(b) Now consider the problem involving two media (see the figure) : the medium I (II), filling the half space $z < 0$ (half space $z > 0$), has appropriate constants ϵ and μ (ϵ' and μ'). When the incident wave is $\vec{E}_{\text{inc}} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$, what are the expected forms for the electric and magnetic fields in the respective medium? [8pts]



(c) What boundary conditions should the fields satisfy at the interface and why? [10pts]

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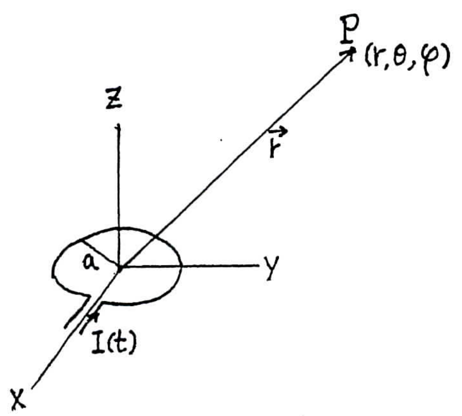
(d) Deduce from the boundary conditions the well-known laws of reflectioin and refraction (i.e., $i = r'$ and $\frac{\sin i}{\sin r} = \frac{\sqrt{\mu'\epsilon'}}{\sqrt{\mu\epsilon}}$, for the angles defined as in the figure.) [12pts]

3. Consider a small circular loop of wire of radius a carrying an oscillating current $I(t) = I_0 \cos \omega t$. The loop is located in the xy plane, with its center at the origin (see the figure).



(a) At not-too-far position $\vec{r}$ such that $a \ll r \ll \frac{c}{\omega}$ (c : the speed of light), only the magnetic field is significant. Find the magnetic field $\vec{B}(\vec{r}, t)$ at such a position. [10pts]

(b) Now, at large distance such that $\frac{r}{c} \gg \frac{1}{\omega}$, obtain the appropriate expressions for the electric and magnetic fields, $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$. [12pts]

(c) Using the results of (b), find the radiated power per unit solid angle. [10pts]

(d) What is the total radiated power of this current loop? [8pts]

(*Hint*) The vector potential due to a static magnetic dipole $\vec{m}$ (at the origin) is $\vec{A}(\vec{r}) = \frac{\vec{m} \times \vec{r}}{r^3}$. On the other hand, the Lorentz-gauge vector potential due to an oscillating magnetic dipole $\vec{m}(t) = \mathrm{Re}(\vec{m}e^{-i\omega t})$, located at the origin, is described in the wave zone by the form (taking the real part understood) $\vec{A}(\vec{r}, t) = ik(\hat{n} \times \vec{m})\frac{e^{ikr - i\omega t}}{r}$, where $\hat{n} = \frac{\vec{r}}{r}$ and $k = \frac{\omega}{c}$.

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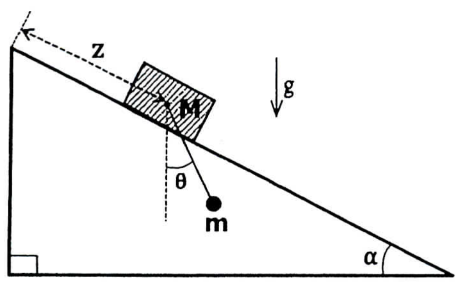
과 목 명 : Classical Mechanics 2010 . 12. 22 시 행

1. A simple pendulum of mass m and length l is attached to a block of mass M which is free to slide down a frictionless plane at an angle α as shown in the figure. The gravitational acceleration is g.



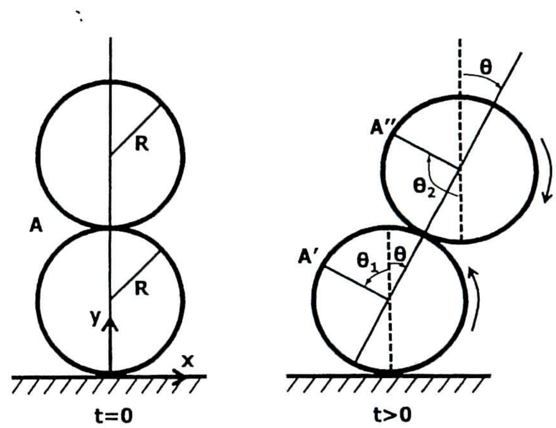
(a) Set up the Lagrangian in terms of z and θ. [8pts]

(b) Find Lagrange's equations of motion for z and θ. [7pts]

(c) The block undergoes an uniform acceleration if the pendulum stays at a certain angle $\theta = \theta_0$ (without jittering). What is the value of θ_0 ? [7pts]

(d) Find $z(t)$ and $\theta(t)$ when the pendulum is under small oscillation about the angle θ_0 (*i.e.*, $\theta(t) = \theta_0 + \delta(t)$, and $\delta(t)$ is small). [8pts]

2. Let us consider a uniform solid cylinder of radius R and mass M. This cylinder rests on a horizontal plane. An identical cylinder rests on top of it when $t < 0$. At $t = 0$, the upper cylinder is given an infinitestimal displacement so that *both* cylinders roll without slipping as in the figure.



(a) Obtain the relationship between the angles θ, θ_1 and θ_2 given in the figure. Note that θ_1 (θ_2) refer to the angle of rotation from the vertical for the bottom (top) cylinder. [7pts]

(b) Obtain the Lagrangian in terms of θ and θ_1. [8pts]

(c) Obtain the Lagrange's equations of motion. What is the energy function? Find other constant(s) of motion. [8pts]

(d) When the cylinders remain in contact, express $\dot{\theta}^2$ in terms of θ, M, R, and g. [7pts]