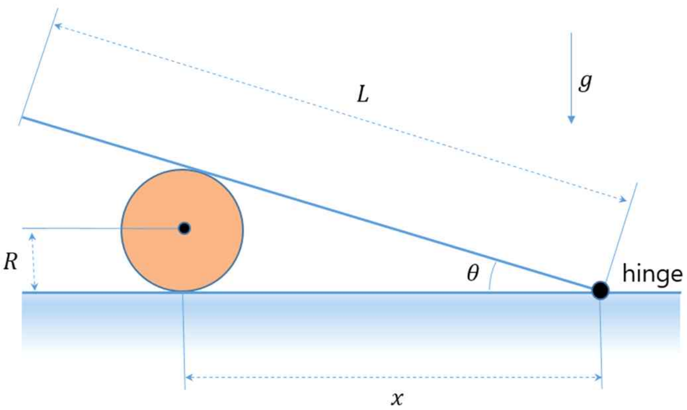
자격시험 문제지

과목명 : 고전역학 **2017 . 07. 18 시행**

1. [30 pts] A cylinder of radius R and uniformly distributed mass m is lying on the ground. A rod of length L and uniform mass M is hinged on the ground (rotating freely around it). The rod leans on the cylinder as shown in the Figure, forming an angle θ with the ground. The distance between the hinge and the bottom of the cylinder is $x\ (< L)$. [Note that, $\tan(\theta/2) = R/x$.] There are no frictions in this system, and the gravitational acceleration is g. The whole system is <u>initially at rest</u>.



(a) [5 pts] Let us denote by N_1, N_2 the normal force on the cylinder by the ground and by the rod, respectively. Show that $N_1 - mg = N_2 \cos\theta$.

(b) [10 pts] Compute the net force F on the cylinder, in terms of N_2 and θ. Also, compute the torque τ on the rod around the hinge, in terms of N_2 , θ and other constants introduced above.

(c) [10 pts] Show that F and τ are related to each other, at the initial time, by

$$F = \frac{2mR}{ML^2 \sin^2(\theta/2)} \tau \quad .$$

(d) [5 pts] Compute N_2 at the initial time, when $\theta = \frac{\pi}{3}$ using the results from (b) and (c).

2. [30 pts] Consider two interacting charged particles with same mass m. The particles are moving in the xy−plane, perpendicular to a uniform magnetostatic field $\vec{B} = B\hat{z}$ where B > 0. Neglect radiation and gravity.

(a) [15 pts] Suppose the charges are identical ($q_1 = q_2 = q$). Find the equations of motion for centre−of−mass (CM) and relative motions separately. If the charges have opposite sign, show the equations of motion do not separate into equations for CM and relative motions.

(b) [8 pts] Express the equation of motion for relative motion from (a) in terms of plane polar coordination (r, θ), in the case of identical charges. Show the $L_w \equiv \mu r^2\left(\dot{\theta} + \frac{1}{2}\omega\right)$ is constant ($w = \frac{qB}{m}$).

(c) [7 pts] If the equation is rewritten for the effective potential $V_e(r)$, show that the motion is always bounded in the presence of a magnetic field.

자격시험 문제지

과목명 : 양자역학 **2017 . 07. 18 시행**

1. [30 pts] Consider *monoenergetic* beam of spin 1/2 particles of mass m and energy E moving in x direction. The potential energy operator is given by

$$V(x) = V_0 - \gamma B_0 S_z \text{ for x > 0}$$

$$V(x) = 0 \text{ for } x \leq 0$$

where $V_0 > 0$ is a positive constant potential, γ is gyromagnetic ratio, $B_0 > 0$ is magnitude of external magnetic field applied in z direction, and S_z is the z−direction spin operator.

(a) [4 pts] Write the Hamiltonian for the particles in the x > 0 region and sketch the potential energy as a function of x for particles having z−direction spins *up and down, respectively*.

(b) [4 pts] Suppose that the spin of particles coming from $-\infty$ is in *eigenstate of the x−direction spin operator* S_x with eigenvalue $+\hbar/2$ and the energy of each particle is $E > V_0 > 0$. Write down the general eigenstate of such an incoming beam considering spatial and spinor parts.

(c) [5 pts] Write down the general solution of transmitted and reflected beams ($E > V_0 > 0$).

(d) [6 pts] What are the boundary conditions at x=0? Using the boundary conditions, write down the equations that must be satisfied by the amplitudes appearing in parts (b) and (c).

(e) [6 pts] If $E = V_0 > 0$, what is the probability of measuring z direction spin angular momentum $+\hbar/2$ for the transmitted beam?

(f) [5 pts] Now, instead of the incoming beam with the x−direction spin polarization (*i.e.*, eigenstate of S_x), if we start with *unpolarized* incoming beam (but still $E = V_0 > 0$), what is the probability of measuring the z−direction spin angular momentum $+\hbar/2$ for the transmitted beam?

2. [30 pts] Consider that we have a hydrogen atom. The orbital angular momentum of the electron is $l = 1$ (p−state). The total angular momentum operator is defined as the sum of the orbital and spin angular momentum operators: J=L+S where L=r×p. The Pauli spin matrices are given by $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Also, it is known that the time−reversal operator for an electron is given by $T = -i\sigma_y K$ where K is the complex−conjugation operator.

(a) [8 pts] What are the possible eigenstates of the total angular momentum operators J^2 and J_z? In other words, determine all the possible eigenstates of the total (orbital and spin) angular momentum operators of the form $|j, m_j\rangle$. For example, $\left|j = \frac{1}{2}, m_j = +\frac{1}{2}\right\rangle$ is one of such eigenstates.

(b) [8 pts] Express the eigenstate $\left|j = \frac{1}{2}, m_j = +\frac{1}{2}\right\rangle$ in terms of the eigenstates of S_z and L_z operators, i.e., $|m_s, m_l\rangle$ states. Hints: Start from $\left|j = \frac{3}{2}, m_j = +\frac{3}{2}\right\rangle = \left|m_l = +1, m_s = +\frac{1}{2}\right\rangle$ and apply J_- on both sides. By doing this, you will obtain the results for $j = \frac{3}{2}$ states. Then, use the orthonormality condition to obtain the expressions for $j = \frac{1}{2}$ states. You may also use the relations $J_\pm |j, m_j\rangle = \sqrt{j(j+1) - m_j(m_j \pm 1)}\,\hbar\,|j, m_j \pm 1\rangle$ and $J_z |j, m_j\rangle = m_j \hbar |j, m_j\rangle$ without proof.

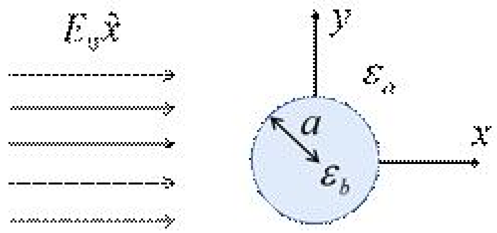
(c) [5 pts] Prove that $T^{-1} = i\sigma_y K$.

(d) [9 pts] Obtain $T L_i T^{-1}$ and $T \sigma_i T^{-1}$ $(i = x, y, z)$. Note that you need to obtain 6 answers. Each correct answer counts 1.5 pts.

자격시험 문제지

과목명 : 전기역학 **2017 . 07. 18 시행**

1. [30 pts] A uniform electric field $E_0\hat{x}$, perhaps produced by means of a parallel plate capacitor, exists in a dielectric having permittivity ϵ_a. With its axis perpendicular to this field, a sufficiently long circular cylindrical dielectric rod [extended along the z (out-of-paper) direction] having permittivity ϵ_b and radius a is introduced.



Because the rod is long and the original field is uniform, the problem reduces down to a two-dimensional polar coordinate problem. Let the axis of cylinder be on the z-axis. (ρ,θ,z) is the cylindrical coordinate. We will look for solutions of Laplace equation inside($\phi_b(\rho,\theta)$) and outside($\phi_a(\rho,\theta)$) of rod subjecting to the boundary conditions

(i) $\phi_a = \phi_b$ $\quad$ at $\rho = a$,

(ii) $\epsilon_b \dfrac{\partial\phi_b}{\partial\rho} = \epsilon_a \dfrac{\partial\phi_a}{\partial\rho}$ $\quad$ at $\rho = a$, and

(iii) $\phi_a \to -E_0 x = -E_0\rho\cos\theta$ $\quad$ for $\rho \gg a$.

(a) [7 pts] Justify boundary conditions (ii) and (iii). You may use the three boundary conditions in solving the next problems even if you haven't solved (a).

(b) [15 pts] The general form of the potential outside and inside the rod satisfying the boundary conditions is given, respectively, by

$$\phi_a = -E_0\rho\cos\theta + \sum_{k=1}^{\infty}\rho^{-k}(C_k\cos k\theta + D_k\sin k\theta)$$

and $\phi_b = \displaystyle\sum_{k=1}^{\infty}\rho^{k}(A_k\cos k\theta + B_k\sin k\theta)$.

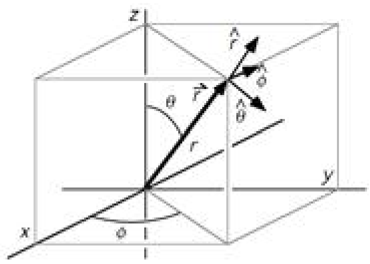
Determine the potential and the electric field at points outside and inside the rod, neglecting end effects.

(c) [8 pts] Sketch the electric field lines inside and outside the rod when $\epsilon_a > \epsilon_b$.

2. [30 pts] The simplest possible spherical electromagnetic wave can be written as

$$\boldsymbol{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}sin(kr-\omega t)\right]\hat{\boldsymbol{\phi}},$$

where $\omega/k = c$.



(a) [10 pts] Obtain the associated magnetic field $\boldsymbol{B}$ using Faraday's law.

(b) [10 pts] Calculate the Poynting vector $\boldsymbol{S} = \dfrac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$.

(c) [10 pts] Take the time average of the Poynting vector to get the intensity vector $\boldsymbol{I} = \langle\boldsymbol{S}\rangle$. Discuss about the results, including the r-dependence.

※ You may use the following formula:

$$\nabla s = \hat{\boldsymbol{r}}\frac{\partial s}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial s}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial s}{\partial\phi}$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\boldsymbol{v} = \hat{\boldsymbol{r}}\frac{1}{r\sin\theta}&\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\\ &+\hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi)\right]\\ &+\hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\end{aligned}$$

※ You may also use the following vacuum Maxwell equation if necessary:

$$\nabla\cdot E = 0,\ \nabla\cdot B = 0,\ \frac{\partial B}{\partial t} = -\nabla\times E,$$

$$\nabla\times B = \frac{1}{c^2}\frac{\partial E}{\partial t}$$

자격시험 문제지

과목명 : 통계역학 **2017 . 07. 18 시행**

1. [60 pts] In a gas of free and independent electrons in three-dimensions, the one-electron levels are specified by the wave vector **k** and spin quantum number s with the energy given by $E(k)=\frac{\hbar^2k^2}{2m}$. At zero temperature, the Fermi energy E_F can be defined in a way that the levels with $E(k)\le E_F$ ($E(k)>E_F$) are occupied (unoccupied).

(a) [5 pts] At the temperature T, the energy density ε and the particle number density n can be written as

$$\epsilon=\frac{2}{V}\sum_k E(k)f(E(k)) \text{ and } n=\frac{2}{V}\sum_k f(E(k))$$

where $f(E)=\frac{1}{e^{(E-\mu)/k_BT}+1}$ is the Fermi-Dirac distribution function with the Boltzmann constant k_B and the chemical potential μ. Using the definition of the density of states $g(E)=\frac{2}{V}\sum_k\delta(E-E(k))$ show that ε and n can be written as

$$\epsilon=\int_{-\infty}^{\infty}dEg(E)Ef(E) \text{ and } n=\int_{-\infty}^{\infty}dEg(E)f(E).$$

(b) [10 pts] Compute $g(E)$ explicitly and show $g(E)=\frac{m}{\hbar^2\pi^2}\sqrt{\frac{2mE}{\hbar^2}}=\frac{3}{2}\frac{n}{E_F}(\frac{E}{E_F})^{1/2}$.

(c) [10 pts] In metals, when the temperature T is much smaller than the chemical potential μ, one can perform the above energy integral by using the Sommerfeld expansion, which leads to

$$\epsilon=\int_0^{\mu}dEg(E)E+\frac{\pi^2}{6}(k_BT)^2[\mu g'(\mu)+g(\mu)]+O(T^4)$$

$$n=\int_0^{\mu}dEg(E)+\frac{\pi^2}{6}(k_BT)^2g'(\mu)+O(T^4).$$

Assuming that the chemical potential μ differs from its $T=0$ value E_F by terms of order T^2, one can write $\int_0^{\mu}dEh(E)=\int_0^{E_F}dEh(E)+(\mu-E_F)h(E_F)$ where $h(E)$ is an arbitrary function. Applying this formula show that $\mu=E_F-\frac{\pi^2}{6}(k_BT)^2\frac{g'(E_F)}{g(E_F)}$ and $\epsilon=\epsilon_0+\frac{\pi^2}{6}(k_BT)^2g(E_F)$ where ε_0 is the energy density in the ground state.

(d) [5 pts] Show that the specific heat of the free electron gas is $c_v=\frac{\pi^2}{2}(\frac{k_BT}{E_F})nk_B$.

(e) [10 pts] To compute the heat capacity $C=\gamma T$, the specific heat should be multiplied by the volume of the system. Suppose that a mole of free electron metal contains ZN_A conduction electrons (Z is the valence and N_A is Avogadro's number). Calculate γ using $R=k_BN_A=8.314$ joules/(mole K)=1.99 calories/(mole K) and $E_F/k_B=10000$ K and $Z=3$. Compute γ in units of J/(mol K) and round up to two significant figures.

(f) [5 pts] In the presence of external magnetic field H, suppose that the energy of an electron with the momentum k whose spin is parallel (antiparallel) to H is given by $E_+(k)=E(k)-\mu_BH$ $[E_-(k)=E(k)+\mu_BH]$. What is the relevant g-factor?

(g) [5 pt] Since the magnetic field induces just a constant shift of single particle energies, one can define the density of states for electrons with the spin parallel (antiparallel) to H as $g_+(E)=\frac{1}{2}g(E-\mu_BH)(g_-(E)=\frac{1}{2}g(E+\mu_BH))$, and the particle density of each species is given by $n_\pm=\int dEg_\pm(E)f(E)$ satisfying $n=n_++n_-$. When the Zeeman energy is much smaller than the Fermi energy, one can assume that $g_\pm(E)=\frac{1}{2}g(E\pm\mu_BH)=\frac{1}{2}g(E)\pm\frac{1}{2}\mu_BHg'(E)$. Using this approximation, find the chemical potential of the system. (Note the the chemical potential is μ when H=0.)

(h) [10 pts] The magnetization density is given by $M=-\mu_B(n_+-n_-)$. Here one can again assume

$$g_\pm(E)=\frac{1}{2}g(E\pm\mu_BH)=\frac{1}{2}g(E)\pm\frac{1}{2}\mu_BHg'(E)$$

Show that the zero temperature magnetic susceptibility is $\chi=\frac{M}{H}=\mu_B^2g(E_F)$.